

## LAB Exercise #4

**The Traction Vector and Stress Tensor**

(Special Thanks to D.D. Pollard who pioneered this exercise in 1991)

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**Introduction**

Stress concentrations in the earth lead to the development of structures such as faults, folds, and joints. This problem set is designed to be a tutorial for stress analysis at a point in a rock mass or throughout a region of homogeneous stress. Of course the state of stress can, and generally does vary throughout a deforming rock mass, but these spatial and temporal variations will not be emphasized. Here we will deal only with the **stresses acting in two dimensions**, that is, in an imaginary plane cutting through a rock mass. The principles you will learn are easily extended to three dimensions.

**Format of lab**

We will lead you through the problems step-by-step, and provide many ways for you to check your work. The objective is to really understand stress analysis. **Show all your work, including visual aids such as figures and printouts from any computer programs you use**, so you can go back to your work at any point and understand what you did. Label all figures and diagrams clearly and profusely. Get help if something is not clear to you. For this exercise, you will be doing work both by hand and with Excel. Work through the problems in the given order to minimize confusion. When you are done, you should understand:

- 1) The difference between the **stress tensor** at a point and the **traction vector** acting on some given plane through that point (see Box 7.3, pg. 184 in T&M for a good explanation of the difference between a vector and a tensor);
- 2) How to calculate the principal stress magnitudes and stress trajectories at a point. Stress trajectories refer to the varying stress orientation in space; and
- 3) What the Mohr circle represents and how to plot a Mohr circle given some stress state.

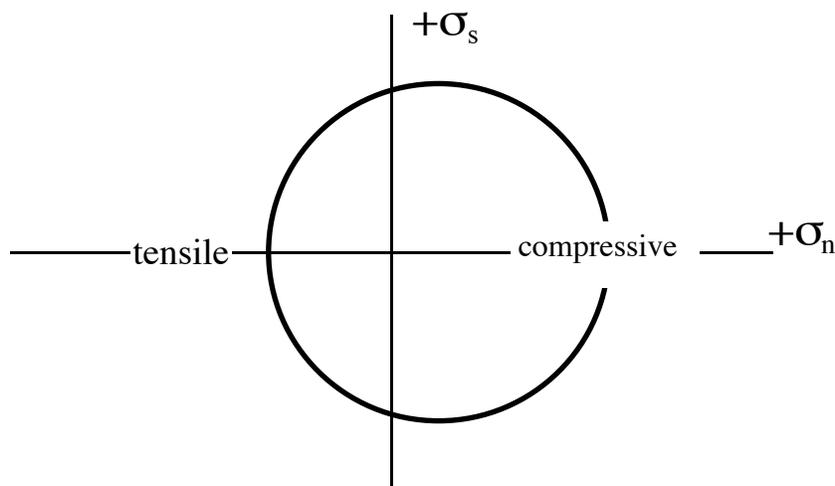
**Preparation reading**

Please refer back to Chapter 7 in Twiss and Moores before you start the problem set. The exercises look very long, but they should go quickly because of all the hints.

**Part One (30 pts) Complete in class**

Use graph paper for diagram and to answer the questions as accurate as you can. For this set of exercises **compressive stresses are positive and tensile stresses are negative**. Therefore,  $\sigma_1$  is the maximum compressive stress (or minimum tensile stress if it happens that all normal stresses are tensile and therefore negative), and  $\sigma_3$  is the minimum compressive stress (or maximum tensile stress).

Recall that  $\sigma_n$  and  $\sigma_s$  are the normal and shear components of the traction vector and plot your Mohr circle with the axes for  $\sigma_n$  and  $\sigma_s$  arranged as shown:



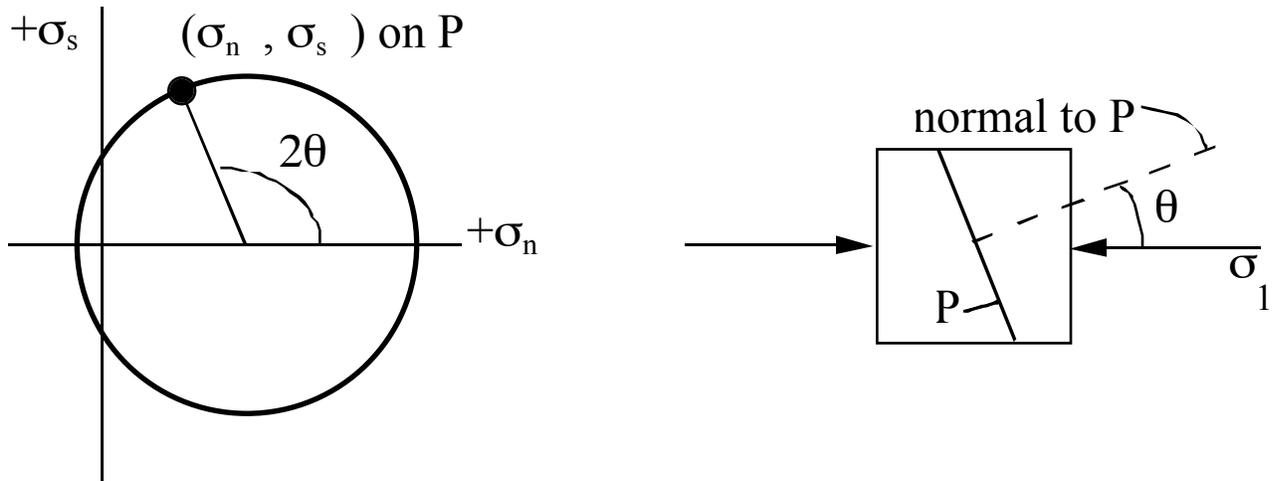
**1a** (5 pt) Two principal stresses acting in a plane at a point are  $\sigma_1 = 60$  MPa and  $\sigma_3 = 20$  MPa. Draw a Mohr circle for this stress state.

**1b** (5 pt) Show where the traction vector components acting on a plane with  $2\theta = 120^\circ$  plots on the Mohr circle. Also show where the components acting on a plane with  $2\theta = 240^\circ$  plot on the Mohr circle.

Remember that the angle  $2\theta$  is measured counterclockwise from the positive  $\sigma_n$  axis.

**1c** (6 pts) Assume that this two dimensional stress state describes the traction vectors acting in a horizontal plane in a certain region of the earth's lithosphere and that the two planes given in (1b) are potential fault planes with vertical dips in this same region. Draw a 2D diagram showing the faults and the principle stresses. Label the faults A and B. Assume that  $\sigma_1$  acts in a north-south direction. Also show the sense of relative motion that the faults would have if they moved (use right-lateral or left-lateral pairs of arrows). The sign conventions shown in problem (2c) may be helpful! On your map be sure to indicate the strike and for each fault.

Remember that in physical space (your map),  $\theta$  is measured counterclockwise from  $\sigma_1$  to the normal for the plane of interest. In stress space,  $2\theta$  is measured counterclockwise from  $(\sigma_n, 0)$  on the Mohr circle to the point on the circle that represents the magnitudes of the pair  $(\sigma_n, \sigma_s)$  acting on a given plane. The following sketch illustrates these relationships.



**1d** (3 pts) What is the angle between  $\sigma_1$  and each of the fault planes? (Use your map)

**1e** (5 pts) Use your Mohr circle to estimate  $(\sigma_n, \sigma_s)$  for each fault plane. Pay attention to sign conventions.

**1f** (6 pts) Confirm your results in part (1e) with the following equations for stress resolution and report your calculated result.

$$\sigma_n = \left[ \frac{\hat{\sigma}_1 + \hat{\sigma}_3}{2} \right] + \left[ \frac{\hat{\sigma}_1 - \hat{\sigma}_3}{2} \right] \cos 2\theta_2$$

$$\sigma_s = \left[ \frac{\hat{\sigma}_1 - \hat{\sigma}_3}{2} \right] \sin 2\theta_2$$

**Part Two (34 pts)**

Answer the questions in part 2 of this exercise based on the following three components of the two-dimensional stress tensor listed below. Recall that the first subscript indicates the normal to the plane on which the stress acts and the second subscript indicates the direction in which the stress acts.

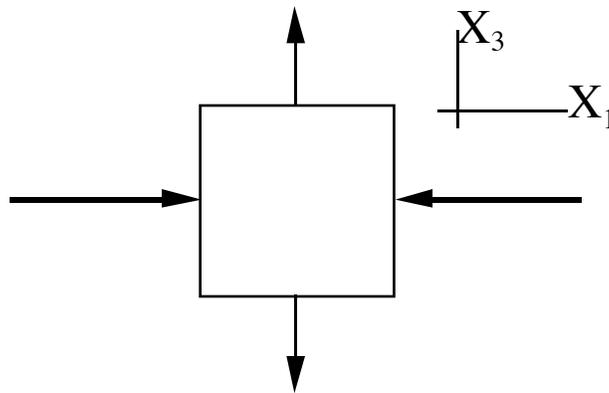
$$\sigma_{11} = 50 \text{ MPa}$$

$$\sigma_{13} = \sigma_{31} = -40 \text{ Mpa}$$

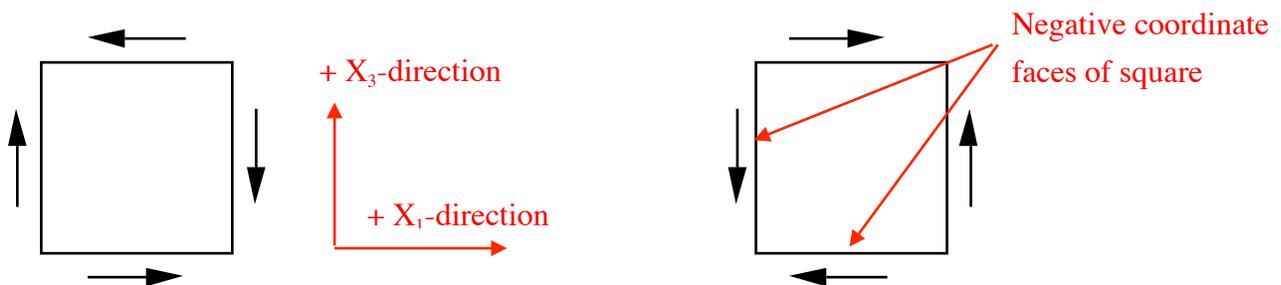
$$\sigma_{33} = 20 \text{ MPa}$$

**2a (4 pts)** Sketch a picture of this stress state (write down the 2D tensor matrix as well). Draw a square "element" and add arrows for  $\sigma_{11}$ ,  $\sigma_{33}$ , and  $\sigma_{13}$ . Make the lengths of your arrows scale (roughly) with the stress magnitudes. Use the following sign conventions: Positive normal stresses are compressive; negative normal stresses are tensile.

For example, the stress state  $\sigma_{11} = +60 \text{ MPa}$  ;  $\sigma_{33} = -30 \text{ MPa}$  would be drawn as follows:



A positive shear stress state is drawn on the left and a negative shear stress is drawn on the right. The sign is determined by the direction of the traction acting on the NEGATIVE coordinate face of the square. Does that make sense?

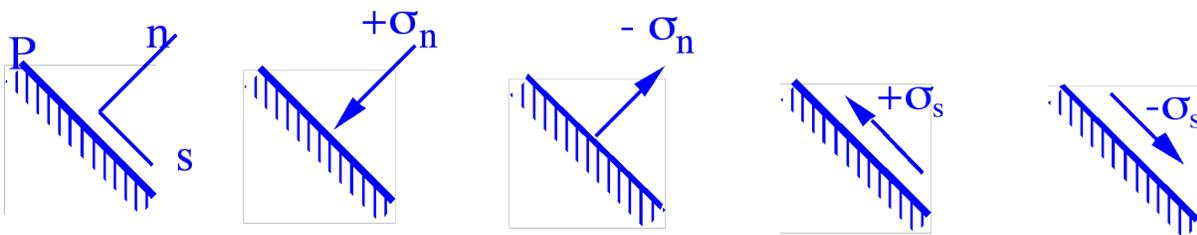


**2b** (3pts) Working with the diagram made in part 2a, add arrows for  $\sigma_1$  and  $\sigma_3$  to your sketch in part 2a. You can tell approximately where the principal stresses are located just from looking at your sketch because  $\sigma_3$  "comes out of the corner that the shear stresses point to." Remember that  $\sigma_1$  and  $\sigma_3$  act in directions that are perpendicular to each other. Your sketch gives you the approximate quadrants for  $\sigma_1$  and  $\sigma_3$ , the exact angles depend on the magnitudes of all three tensor components. You will calculate the exact angles in part (2e).

**2c** (5 pts) Construct a Mohr circle diagram for this stress state. Use graph paper and label the points that you use to draw the circle.

HINT: To construct the circle, plot  $(\sigma_n, \sigma_s)$  for a plane orthogonal to  $X_1$  and for a plane orthogonal to  $X_3$ , i.e. the 1 and 3 planes respectively. These two points will plot on opposite ends of a diameter passing through the center (a point on the  $\sigma_n$  axis) of the Mohr circle. After plotting these points, just draw a circle through the two points with the appropriate center and radius. (A compass will come in handy.)

**2d** (3 pts) Based on your Mohr circle, what are the approximate magnitudes of  $\sigma_1$  and  $\sigma_3$ ? The sign conventions for the components of the traction vector acting on a plane, P, are as follows:



Note that the tractions are positive when they act in the negative coordinate directions. The coordinates  $(\mathbf{n}, \mathbf{s})$  are always drawn such that  $\mathbf{n}$  is positive outward and  $\mathbf{s}$  is directed to the right when facing in a positive  $\mathbf{n}$  direction. Look at your sketch in part (2a) to determine the appropriate signs for  $\sigma_s$  on planes parallel to the  $X_1$  and  $X_3$  axes. Note the possible difference between the signs of the shear traction vector component and the shear stress tensor component. Clearly, one must be careful to state which component you are talking about.

**2e** (6 pts) Compute the principal stress magnitudes using:

$$\sigma_1 = \frac{1}{2}(\sigma_{11} + \sigma_{33}) + \left[ \frac{1}{4}(\sigma_{11} - \sigma_{33})^2 + (\sigma_{13})^2 \right]^{\frac{1}{2}}$$

$$\sigma_3 = \frac{1}{2}(\sigma_{11} + \sigma_{33}) - \left[ \frac{1}{4}(\sigma_{11} - \sigma_{33})^2 + (\sigma_{13})^2 \right]^{\frac{1}{2}}$$

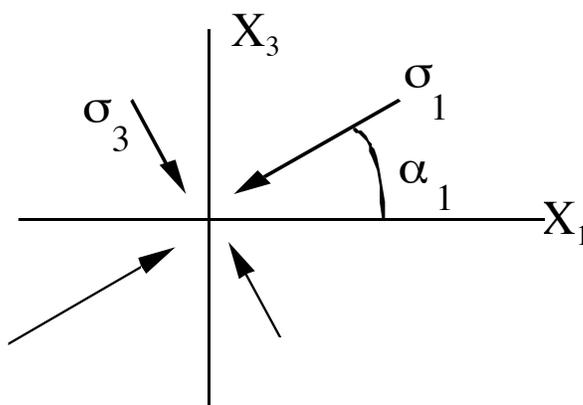
Show your work. Check to be sure that your results are consistent with your Mohr circle in (2c).

**2f** (6 pts) Compute the principal stress directions using:

$$\alpha_1 = \frac{1}{2} \tan^{-1} \left( \frac{2\sigma_{13}}{\sigma_{11} - \sigma_{33}} \right)$$

$$\alpha_3 = \alpha_1 + 90^\circ$$

Remember that  $\alpha_1$  and  $\alpha_3$  are the angles between the  $X_1$ -axis and the directions in which  $\sigma_1$  and  $\sigma_3$  act, respectively. They are measured with a positive sense being counterclockwise. Do not confuse the angle  $\alpha$  with the angle  $\theta$ . See the picture below and refer to the picture on page 3 for clarification of this point.



Be sure that your angles agree with your picture in part 2a; your sketch will help you decide which quadrant is correct for  $\sigma_1$  and  $\sigma_3$ . The following hint explains how to use the equations to get the correct angles.

**HINT:** Most hand calculators will return an angle between  $-90^\circ$  and  $+90^\circ$  for the arctan function. Therefore, you may have to add some multiple of  $90^\circ$  to achieve the desired result. You must consider the sign of the numerator and the sign of the denominator in the fraction that is the argument for the arctan function as indicated below:

**Case 1:** if  $\sigma_{13}$  is positive and  $(\sigma_{11} - \sigma_{33})$  is positive, then the *arctan* is between  $0^\circ$  and  $90^\circ$ .

**Case 2:** if  $\sigma_{13}$  is positive and  $(\sigma_{11} - \sigma_{33})$  is negative, then the *arctan* is between  $90^\circ$  and  $180^\circ$ .

**Case 3:** if  $\sigma_{13}$  is negative and  $(\sigma_{11} - \sigma_{33})$  is negative, then the *arctan* is between  $180^\circ$  and  $270^\circ$ .

**Case 4:** if  $\sigma_{13}$  is negative and  $(\sigma_{11} - \sigma_{33})$  is positive, then the *arctan* is between  $270^\circ$  and  $360^\circ$ .

This procedure should remind you of the "unit circle" and the definition of the tangent function from your high school math class. You would be surprised to know how many graduate students and professors botch up this calculation! **Also, remember the factor of one-half in the formula; find the correct arctan before you multiply by 0.5 to solve for  $\alpha_1$ .** Of course with modern computers all of these calculations can be done for you in microseconds, but **you must understand the underlying principles and use the correct function!**

Use **Mohr\_Stress.xls** to complete question 2g.

Download the excel file **Mohr\_Stress.xls** from the website. On a PC, click the right mouse button and choose "Save Link to Disk" or something similar (varies by browser). On a Mac, click and hold the mouse button until a menu pops up. Choose "Save Link to Disk."

Open the file. The main page is to input the Stress values. In the two additional pages is the Mohr Circle figure and Traction Component figure.

The input parameters for this application are the stresses  $\sigma_{11}$ ,  $\sigma_{33}$ , and  $\sigma_{13}$ ; these stresses completely determine the stress state in two dimensions. You can change each of the values for these three input stresses, pressing the "enter" key after each input. Excel then computes the principal stress magnitudes and directions. Excel also draws a Mohr circle in the tab Mohr Circle.

CAUTION -- You may have to "grab" the Mohr circle chart by the lower right corner and stretch the plot into a circular shape because the scale changes every time you change the input parameters.

Change the input parameters to be  $\sigma_{11} = 50$  MPa,  $\sigma_{13} = -40$  MPa,  $\sigma_{33} = 20$  MPa. Check to see that Excel's computed principal stress magnitudes, principal stress directions, and Mohr circle agree with your results in parts (a) - (d).

**2g** (7 pts) Assume that this stress state represents the stress tensor components acting in a horizontal plane in some region of the earth and that the  $X_3$ -axis is north. If vertical joints (opening mode fractures) formed in this region, what would their strike and dip be? Draw a map showing some of the joints, the principal stresses, and a north arrow.

HINT: Assume that the joint faces form perpendicular to the least compressive stress acting in the horizontal plane.

Be sure that you understand the difference between the stress tensor at a point and the traction vector acting on some given plane. The Mohr circle is a way of plotting the normal and shear components for traction vectors associated with all possible planes through a point. In part **3** of this exercise we will study a different way to represent the stress state.

Use **Mohr\_Stress.xls** to complete Part Three and Part Four

### Part Three (21 pts)

Another representation of the state of stress at a point is found by plotting  $\theta$  vs.  $\sigma_n$  and  $\theta$  vs.  $\sigma_s$ . Change the input parameters to be  $\sigma_{11} = 60$  MPa,  $\sigma_{13} = 0$  MPa, and  $\sigma_{33} = 20$  MPa. Notice how the plots have changed. Use the chart Traction Components to answer the following questions:

**3a** (3 pt) What is the magnitude of the shear stress on planes that carry the maximum and minimum normal stresses?

**3b** (3 pt) What is special about the magnitude of the normal stress on planes that carry the maximum shear stress?

**3c** (5pts) Get a printout of the chart and show that the planes with maximum shear stress are  $45^\circ$  away from planes of maximum normal stress. Show (with a sketch) how this relationship can also be seen on the Mohr circle.

**3d** (5 pts) Show on your plot of  $\sigma_n$  that the principal stresses act on planes that are  $90^\circ$  apart and show the value of the mean normal stress.

**3e** (5 pts) What are two different ways to completely specify the state of two dimensional stress at a point? What is the minimum amount of information (how many numbers do you need)?

### Part Four (15 pts)

For part 4 of this exercise you are given the following components of the stress tensor:

$$\sigma_{11} = 30 \text{ MPa} = \sigma_3$$

$$\sigma_{13} = 0 \text{ MPa}$$

$$\sigma_{33} = 40 \text{ MPa} = \sigma_1$$

Suppose that this is the state of stress on a vertical cross-section at some depth in the earth with  $X_3$  pointing upward.

**4a** (5 pts) Use Excel to find the shear stress and normal stress on planes with normals oriented at  $30^\circ$  and  $60^\circ$  from the  $X_3$ -axis.

**4b** (5 pts) Get a printout of the Mohr circle and show where the stress components acting on these two planes plot on the circle.

**4c** (5 pts) If these two planes represent potential fault surfaces, which one is more likely to fail?

HINT: Assume faulting is a frictional process driven by shear stress and resisted by compressive stress.