
LAB Exercise #4 - Answers

The Traction Vector and Stress Tensor

Due: Thursday, 26 February 2009

(Special Thanks to D.D. Pollard who pioneered this exercise in 1991)

Introduction

Stress concentrations in the earth lead to the development of structures such as faults, folds, and joints. This problem set is designed to be a tutorial for stress analysis at a point in a rock mass or throughout a region of homogeneous stress. Of course the state of stress can, and generally does vary throughout a deforming rock mass, but these spatial and temporal variations will not be emphasized. Here we will deal only with the **stresses acting in two dimensions**, that is, in an imaginary plane cutting through a rock mass. The principles you will learn are easily extended to three dimensions.

Format of lab

We will lead you through the problems step-by-step, and provide many ways for you to check your work. The objective is not just to get the correct answers, but to really understand stress analysis. **Show all your work, including visual aids such as figures and printouts from any computer programs you use**, so you can go back to your work at any point and understand what you did. Label all figures and diagrams clearly and profusely. Get help if something is not clear to you. For this exercise, you will be doing work both by hand and with Excel. Work through the problems in the given order to minimize confusion. When you are done, you should understand:

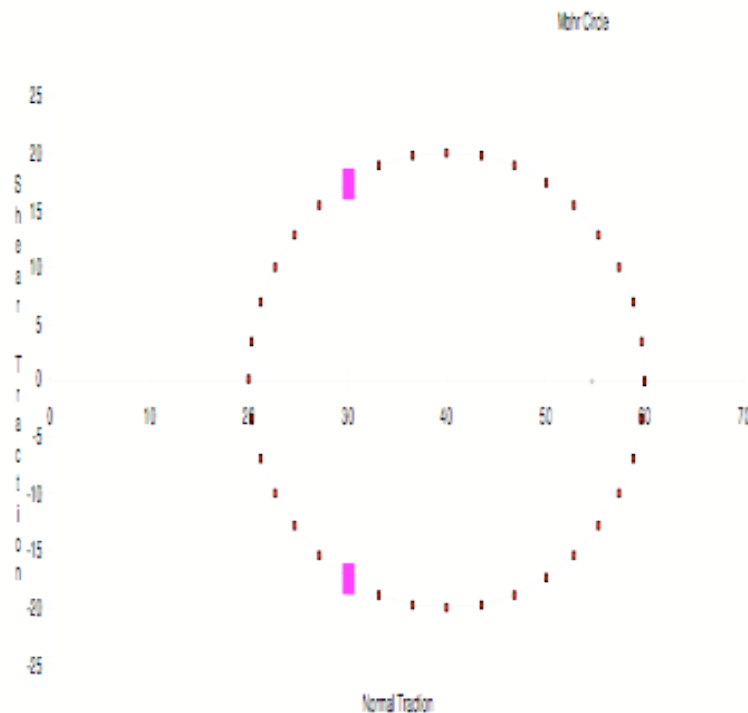
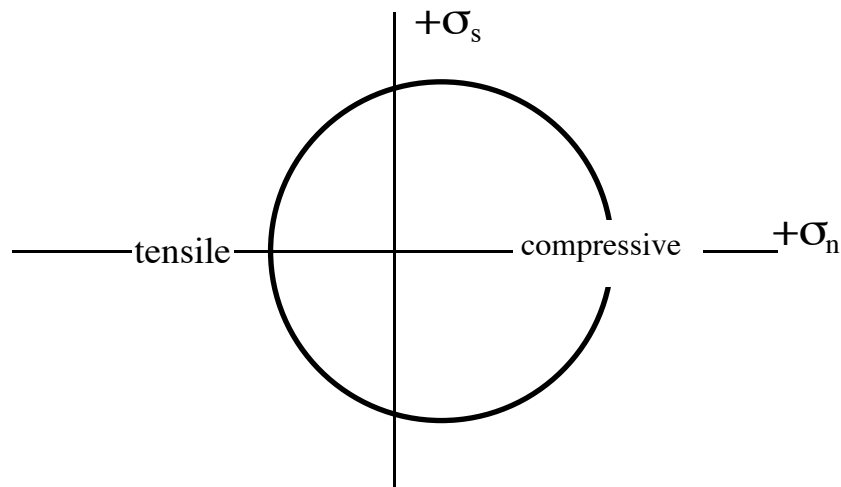
- 1) The difference between the **stress tensor** at a point and the **traction vector** acting on some given plane through that point (see Box 7.3, pg. 184 in T&M for a good explanation of the difference between a vector and a tensor);
- 2) How to calculate the principal stress magnitudes and stress trajectories (are these the principal axes?) at a point; and
- 3) What the Mohr circle represents and how to plot a Mohr circle given some stress state.

Preparation reading

Please refer back to Chapter 7 in Twiss and Moores and, if you would like, read Marshak and Mitra pp. 193-199, before you start the problem set. The exercises look very long, but they should go quickly because of all the hints.

Part One (30 pts)

1a (5 pt) Given that the two principal stresses acting in a plane at a point are $\sigma_1 = 60$ MPa and $\sigma_3 = 20$ MPa, **draw a Mohr circle for this stress state**. Use graph paper and be as accurate as you can. For this set of exercises *compressive stresses are positive and tensile stresses are negative*. Therefore, σ_1 is the maximum compressive stress (or minimum tensile stress if it happens that all normal stresses are tensile and therefore negative), and σ_3 is the minimum compressive stress (or maximum tensile stress). Recall that σ_n and σ_s are the normal and shear components of the traction vector and plot your Mohr circle with the axes for σ_n and σ_s arranged as follows:



-2 if no label for σ_n and σ_s

-2 if no scale bar or labeling on axis

1b (5 pt) Show where the traction vector components acting on a plane with $2\theta = 120^\circ$ plot on your Mohr circle. Also show where the components acting on a plane with $2\theta = 240^\circ$ plot on your Mohr circle. Remember that the angle 2θ is measured counterclockwise from the positive σ_n axis.

1c (6 pts) Assume that this two dimensional stress state describes the traction vectors acting in a horizontal plane in a certain region of the earth's lithosphere and that the two planes given in (1b) are potential fault planes with vertical dips in this same region. Draw a map (3D block diagram would be nice) showing the faults and the principle stresses (assume that σ_1 acts in a north-south direction). Also show the sense of relative motion that the faults would have if they moved (use right-lateral or left-lateral pairs of arrows). The sign conventions shown in problem (2c) may be helpful! On your map be sure to indicate the strike and dip for each fault.

Remember that in physical space (your map), θ is measured counterclockwise from σ_1 to the **normal** for the plane of interest. In stress space, 2θ is measured counterclockwise from $(\sigma_1, 0)$ on the Mohr circle to the point on the circle that represents the magnitudes of the pair (σ_n, σ_s) acting on a given plane. The following sketch illustrates these relationships.

-1 if no orientation

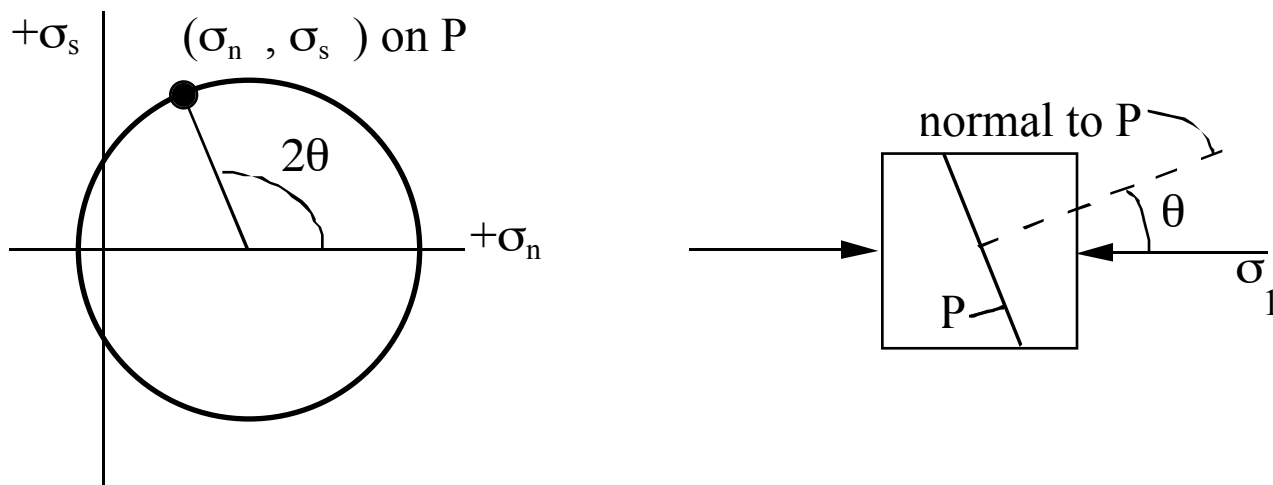
-1 if gave "vertical" for dip instead of 90°

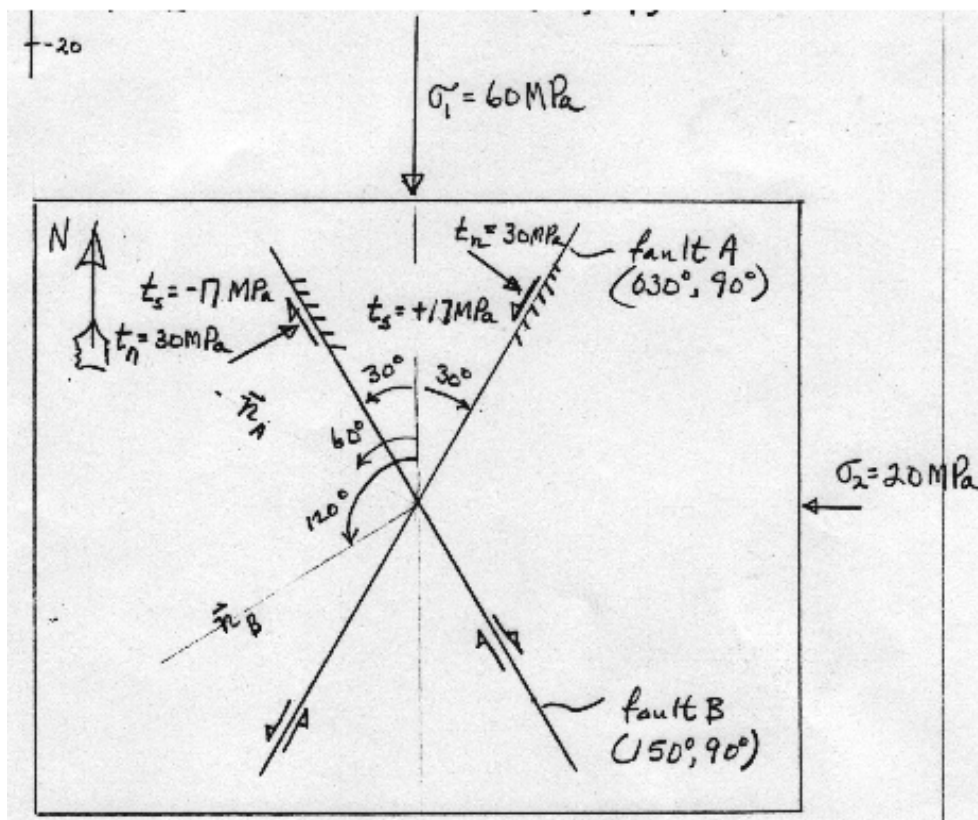
-1 if reversed fault plane strikes

-1 for each wrong sense of slip

-3 if no strike and dip

-1 if didn't scale arrows to magnitude of stresses





1d (3 pts) What is the angle between σ_1 and each of the fault planes? (Use your map)

For $2\theta = 120^\circ$, fault is 30° E of σ_1

For $2\theta = 240^\circ$, fault is 30° W of σ_1

-1 if didn't specify angle on map or give direction from σ_1

1e (5 pts) Use your Mohr circle to estimate (σ_n, σ_s) for each fault plane. Pay attention to sign conventions.

$\sigma_n = 30$; $\sigma_s = \pm 17.3$

-1 if no negative for 240° shear traction

1f (6 pts) Confirm your results in part (1e) with the following equations for stress resolution:

$$\sigma_n = \left[\frac{\hat{\sigma}_1 + \hat{\sigma}_3}{2} \right] + \left[\frac{\hat{\sigma}_1 - \hat{\sigma}_3}{2} \right] \cos 2\theta_2$$

$$\sigma_s = \left[\frac{\hat{\sigma}_1 - \hat{\sigma}_3}{2} \right] \sin 2\theta_2$$

$$\sigma_n = 40 + 20 \cos 120 = 40 - 10 = 30.0 \text{ MPa}; \sigma_s = 20 \sin 120 = 17.3 \text{ MPa}; \sigma_n = 40 + 20 \cos 240 = 40 - 10 = 30.0 \text{ MPa}; \sigma_s = 20 \sin 240 = -17.3 \text{ MPa}$$

Part Two (34 pts)

Answer the questions in **part 2** of this exercise based on the following three components of the *two dimensional stress tensor*:

$$\sigma_{11} = 50 \text{ MPa}$$

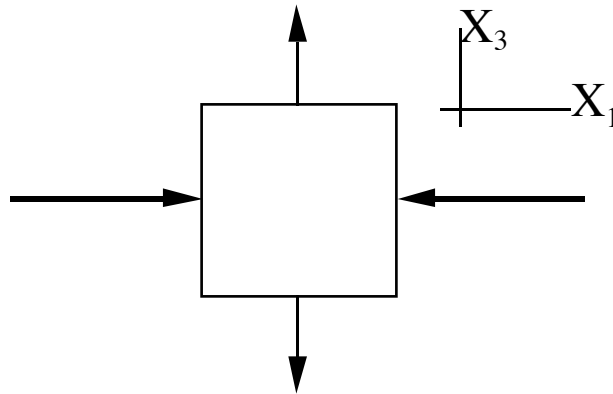
$$\sigma_{13} = \sigma_{31} = -40 \text{ MPa}$$

$$\sigma_{33} = 20 \text{ MPa}$$

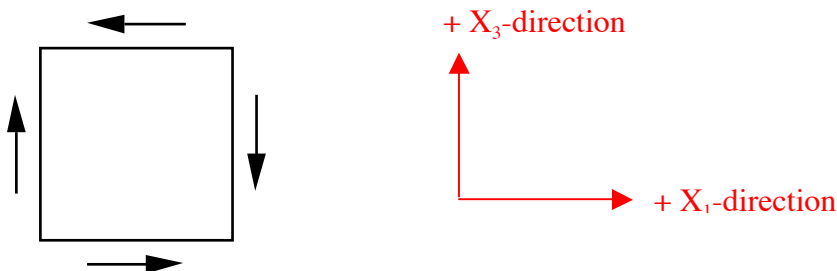
Recall that the first subscript indicates the (normal to the) plane on which the stress acts and the second subscript indicates the direction in which the stress acts.

2a (4 pts) Sketch a picture of this stress state (write down the 2D tensor matrix as well). Draw a square "element" and add arrows for σ_{11} , σ_{33} , and σ_{13} . Make the lengths of your arrows scale (roughly) with the stress magnitudes. Use the following sign conventions:

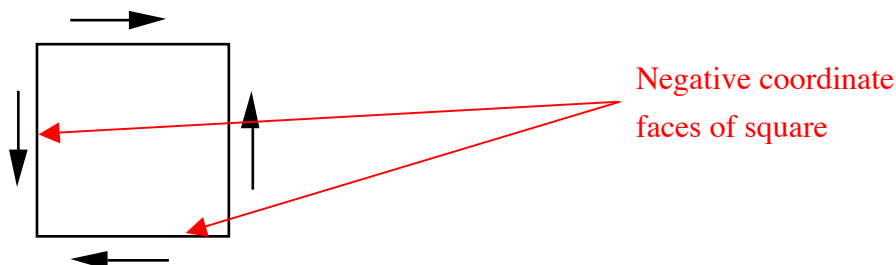
Positive normal stresses are compressive; negative normal stresses are tensile. For example, the stress state $\sigma_{11} = +60 \text{ MPa}$; $\sigma_{33} = -30 \text{ MPa}$ would be drawn as follows:



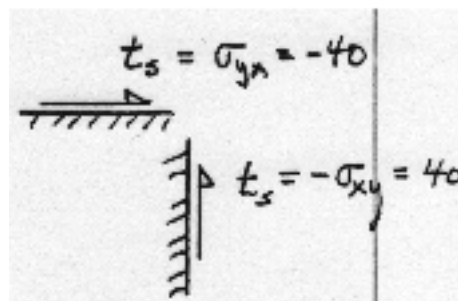
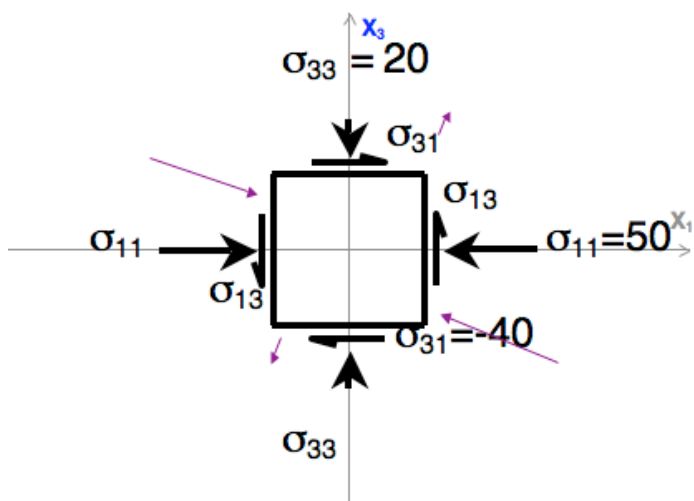
A positive shear stress state is drawn (sign is determined by the direction of the traction acting on the **NEGATIVE** coordinate face of the square...does that make sense?):



whereas a negative shear stress state is drawn:



[50 -40]
[-40 20]



- 1 if tensor is wrong
- 1 if arrows aren't scaled
- 1 for each set of arrows that pointed in wrong direction

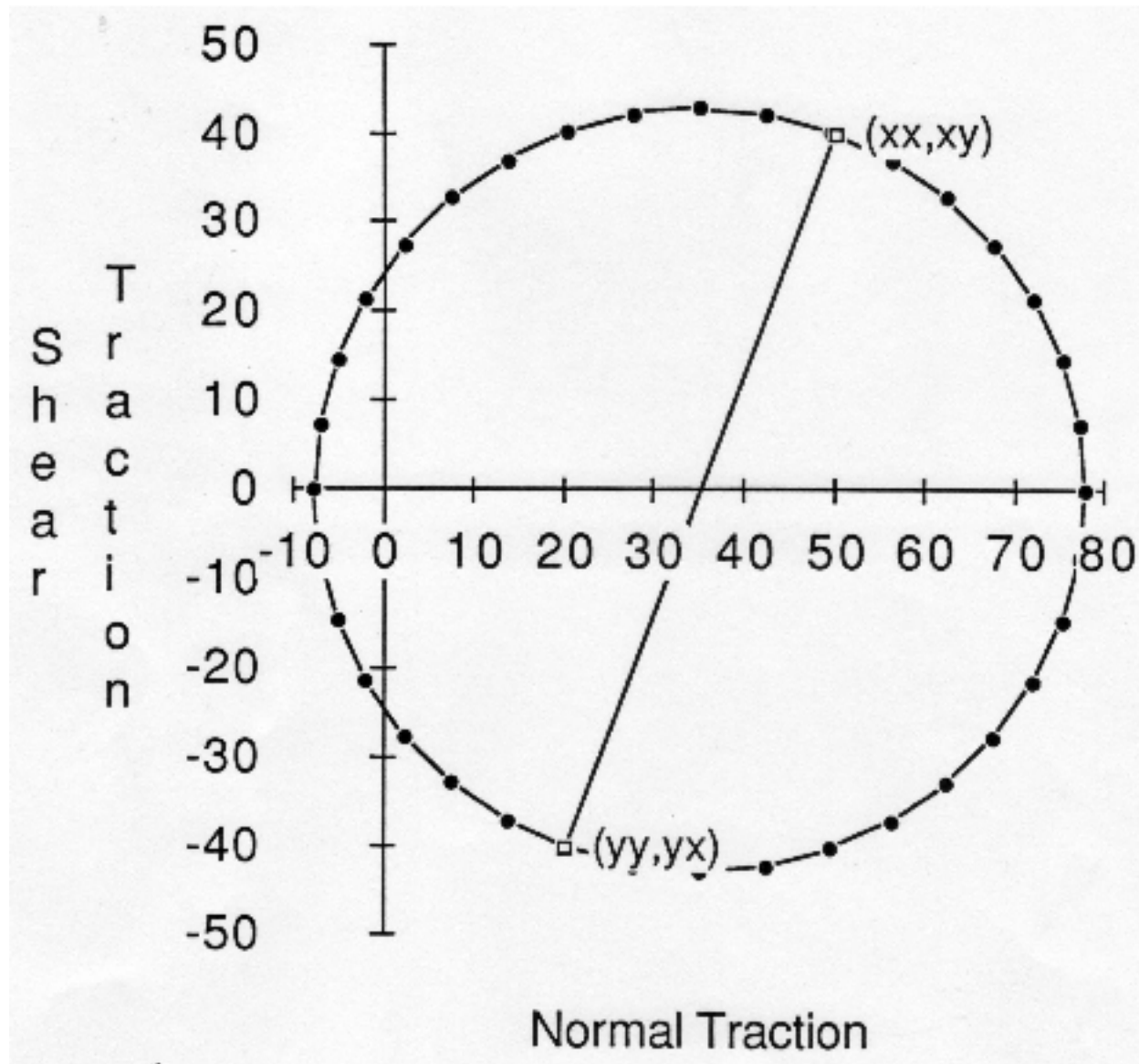
2b (3pts) Now add arrows for σ_1 and σ_3 to your sketch in part (2a). You can tell approximately where the principal stresses are located just from looking at your sketch because σ_3 "comes out of the corner that the shear stresses point to." Remember that σ_1 and σ_3 act in directions that are perpendicular to each other. Your sketch gives you the approximate quadrants for σ_1 and σ_3 , the exact angles depend on the magnitudes of all three tensor components. You will calculate the exact angles in part (2e).

- 1 if oriented in wrong direction

2c (5 pts) Construct a Mohr circle diagram for this stress state. Use graph paper and label the points that you use to draw the circle.

HINT: To construct the circle, plot (σ_n, σ_s) for a plane parallel to

the x-axis and for a plane parallel to the y-axis. These two points will plot on opposite ends of a diameter passing through the center (a point on the σ_n axis) of the Mohr circle. After plotting these points, just draw a circle through the two points with the appropriate center and radius. (A compass will come in handy.)



-1 if didn't label points

-1 if wrong 2 points

-1 if axis isn't labeled

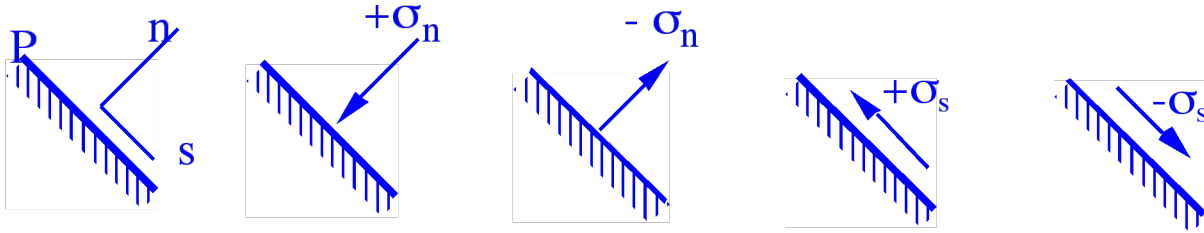
-1 if no scale

2d (3 pts) Based on your Mohr circle, what are the magnitudes of σ_1 and σ_3 (approximately)?

$$\sigma_1 = 77.7 = 0.5 (50 + 20) + [0.25(50-20)^2 + (-40)^2]^{0.5}$$

$$\sigma_2 = -7.7 = 0.5(50 + 20) - [0.25(50 - 20)^2 + (-40)^2]^{0.5}$$

The sign conventions for the components of the traction vector acting on a plane, P, are as follows:



Note that the tractions are positive when they act in the negative coordinate directions. The coordinates (n, s) are always drawn such that n is positive outward and s is directed to the right when facing in a positive n direction. Look at your sketch in part (2a) to determine the appropriate signs for σ_s on planes parallel to the X_1 and X_3 axes. Note the possible difference between the signs of the shear traction vector component and the shear stress tensor component. Clearly, one must be careful to state which component you are talking about.

-1 for each wrong sign

2d (6 pts) Compute the principal stress magnitudes using:

$$\sigma_1 = \frac{1}{2}(\sigma_{11} + \sigma_{33}) + \left[\frac{1}{4}(\sigma_{11} - \sigma_{33})^2 + (\sigma_{13})^2 \right]^{\frac{1}{2}}$$

$$\sigma_3 = \frac{1}{2}(\sigma_{11} + \sigma_{33}) - \left[\frac{1}{4}(\sigma_{11} - \sigma_{33})^2 + (\sigma_{13})^2 \right]^{\frac{1}{2}}$$

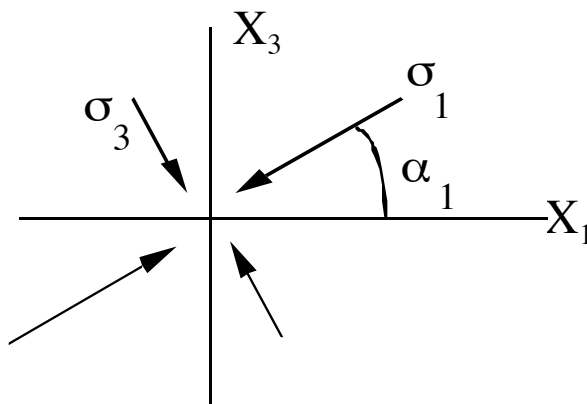
Show your work. Check to be sure that your results are consistent with your Mohr circle in (2c).

2e (6 pts) Compute the principal stress directions using:

$$\alpha_1 = \frac{1}{2} \tan^{-1} \left(\frac{2\sigma_{13}}{\sigma_{11} - \sigma_{33}} \right)$$

$$\alpha_3 = \alpha_1 + 90^\circ$$

Remember that α_1 and α_3 are the angles between the X_1 -axis and the directions in which σ_1 and σ_3 act, respectively. They are measured with a positive sense being counterclockwise. Do not confuse the angle α with the angle θ . See the picture below and refer to the picture on page 3 for clarification of this point.



Be sure that your angles agree with your picture in part (2a); your sketch will help you decide which quadrant is correct for σ_1 and σ_3 . The following hint explains how to use the equations to get the correct angles.

HINT: Most hand calculators will return an angle between -90° and $+90^\circ$ for the *arctan* function. Therefore, you may have to add some multiple of 90° to achieve the desired result. You must consider the sign of the numerator and the sign of the denominator in the fraction that is the argument for the *arctan* function as indicated below:

Case 1: if σ_{13} is positive and $(\sigma_{11} - \sigma_{33})$ is positive, then the *arctan* is between 0° and 90° .

Case 2: if σ_{13} is positive and $(\sigma_{11} - \sigma_{33})$ is negative, then the *arctan* is between 90° and 180° .

Case 3: if σ_{13} is negative and $(\sigma_{11} - \sigma_{33})$ is negative, then the *arctan* is between 180° and 270° .

Case 4: if σ_{13} is negative and $(\sigma_{11} - \sigma_{33})$ is positive, then the *arctan* is between 270° and 360° .

This procedure should remind you of the "unit circle" and the definition of the tangent function from your high school math class. You would be surprised to know how many graduate students and professors botch up this calculation! Also, remember the factor of one-half in the formula; once you find the correct *arctan*, you must divide by two to solve for α_1 . Of course with modern computers all of these calculations can be done for you in microseconds, but **you must understand the underlying principles and use the correct function!**

$$.5 * \text{atan}(2 * .40) / (50 - 20)$$

$$.5 * \text{atan}(-2.66)$$

.5*-69.44

Report angle between 270 and 360: .5 * 290.56

$$\alpha_1 = 145.3^\circ$$

$$\alpha_3 = 145.28 + 90 = 235.3^\circ$$

-2 if thought α_1 had to be between 270°-360°

-3 if didn't attempt to put atan part between 270°-360°

-1 if used wrong α_1 for α_3 calculation

2f Download the excel files from the web site. On a PC, click the right mouse button and choose "Save Link to Disk" or something similar (varies by browser). On a Mac, click and hold the mouse button until a menu pops up. Choose "Save Link to Disk." Double click on the stress folder to open it and shift click on the **stress.w**, **Mohr.c**, and **Sigma.c** files. Open Microsoft Excel. File...Open... You may need to select "All Readable Files" from the File Type drop-down menu in order to be able to see the three files listed above (again, the words vary from version to version). Open up all three of the files.

The input parameters for this application are the stresses σ_{11} , σ_{33} , and σ_{13} ; these stresses completely determine the stress state in two dimensions. You can change each of the values for these three input stresses, pressing the "enter" key after each input. Excel then computes the principal stress magnitudes and directions. Excel also draws a Mohr circle in the chart "**Mohr.c**".

CAUTION -- You may have to "grab" the Mohr circle chart by the lower right corner and stretch the plot into a circular shape because the scale changes every time you change the input parameters.

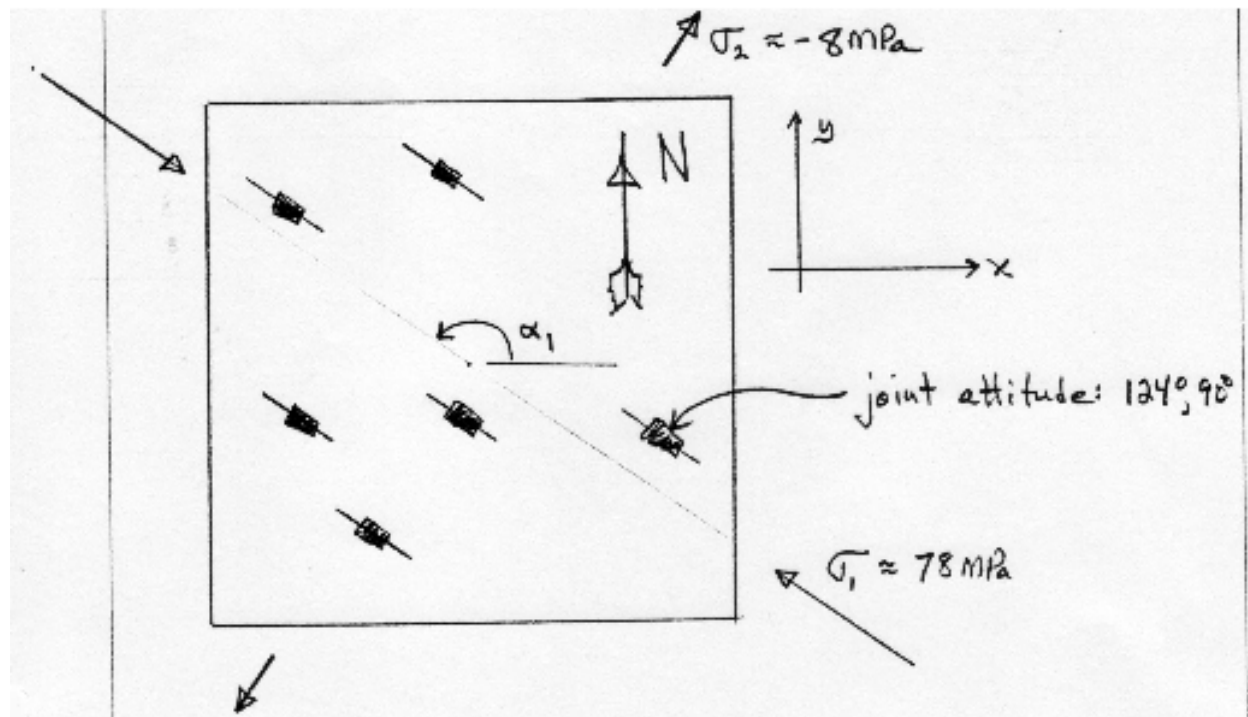
Change the input parameters to be $\sigma_{11} = 50$ MPa, $\sigma_{13} = -40$ MPa, and $\sigma_{33} = 20$ MPa. Check to see that Excel's computed principal stress magnitudes, principal stress directions, and Mohr circle agree with your results in parts (a) - (d). To see the Mohr circle chart, pull down the "window" menu and choose "**Mohr.c**".

2g (7 pts) Assume that this stress state represents the stress tensor components acting in a horizontal plane in some region of the earth and that the X₁-axis is north. If vertical joints (opening mode fractures) formed in this region, what would their strike and dip be? Draw a map showing some of the joints, the principal stresses, and a north arrow.

HINT: Assume that the joint faces form perpendicular to the least compressive stress acting in the horizontal plane.

Be sure that you understand the difference between the stress tensor at a point and the traction vector acting on some given plane. The Mohr circle is a way of plotting the normal and shear components for traction vectors associated with all possible planes

through a point. In part 3 of this exercise we will study a different way to represent the stress state.



$124^\circ, 90^\circ$ dip

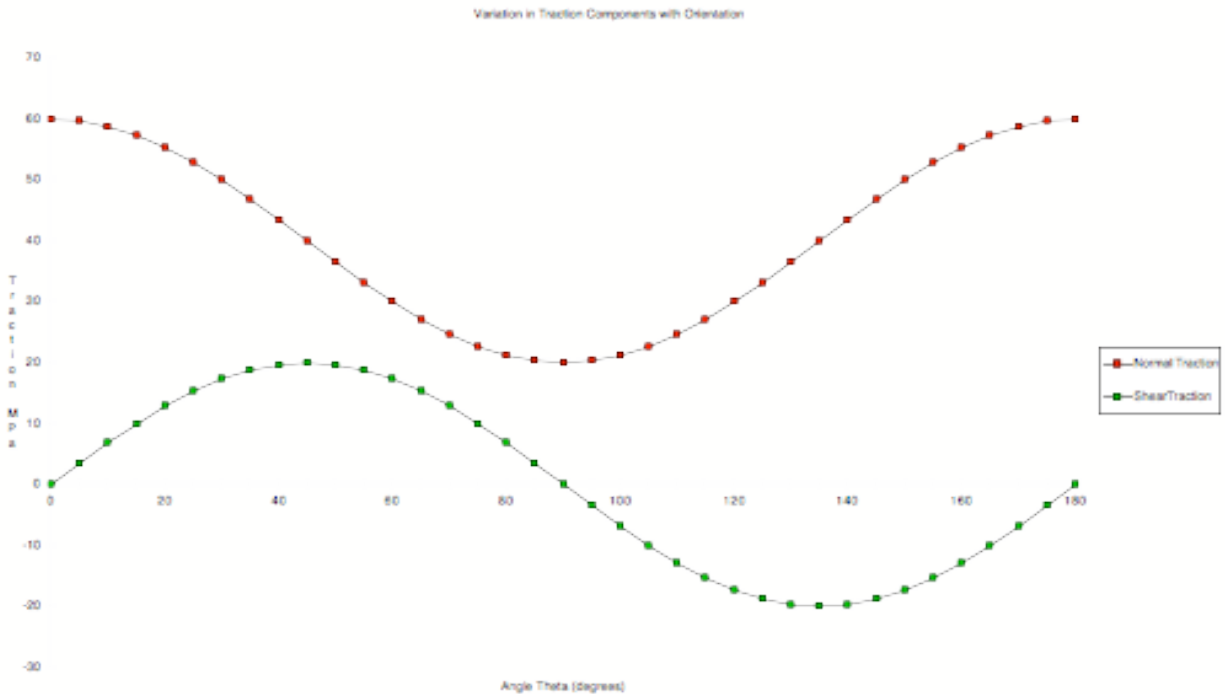
-2 if wrong strike

-2 if not principal stresses

-1 if no north orientation

Part Three (21 pts)

Another representation of the state of stress at a point is found by plotting θ vs. σ_n and θ vs. σ_s . Pull down the "File" menu and choose the chart "Sigma.c". Study the plots. Now go back to the "stress.w" worksheet and change the input parameters to be $\sigma_{11} = 60 \text{ MPa}$, $\sigma_{13} = 0 \text{ MPa}$, and $\sigma_{33} = 20 \text{ MPa}$. Pull down the window menu and choose "Sigma.c" again. Notice how the plots have changed. Use the chart "Sigma.c" to answer the following questions:



3a (3 pt) What is the magnitude of the shear stress on planes that carry the maximum and minimum normal stresses?

zero

3b (3 pt) What is special about the magnitude of the normal stress on planes that carry the maximum shear stress?

It is the mean normal stress

-2 if said $\frac{1}{2}$ max normal stress

-3 if said principal stress

3c (5pts) Get a printout of the chart and show that the planes with maximum shear stress are 45° away from planes of maximum normal stress. Show (with a sketch) how this relationship can also be seen on the Mohr circle.

$2\theta = 90^\circ$ on the Mohr circle so $\theta = 45^\circ$

3d (5 pts) Show on your plot of σ_n that the principal stresses act on planes that are 90° apart and show the value of the mean normal stress. $\sigma_m = 40 \text{ MPa}$

3e (5 pts) What are two different ways to completely specify the state of two dimensional stress at a point? What is the minimum amount of information (how many numbers do you need)?

2 principal stresses and α_1 or 3 components of stress tensor

Part Four (15 pts)

For part 4 of this exercise you are given the following components of the stress tensor:

$$\sigma_{11} = 30 \text{ MPa} = \sigma_{33}$$

$$\sigma_{13} = 0 \text{ MPa}$$

$$\sigma_{33} = 40 \text{ MPa} = \sigma_1$$

Suppose that this is the state of stress on a vertical cross-section at some depth in the earth with X_3 pointing upward.

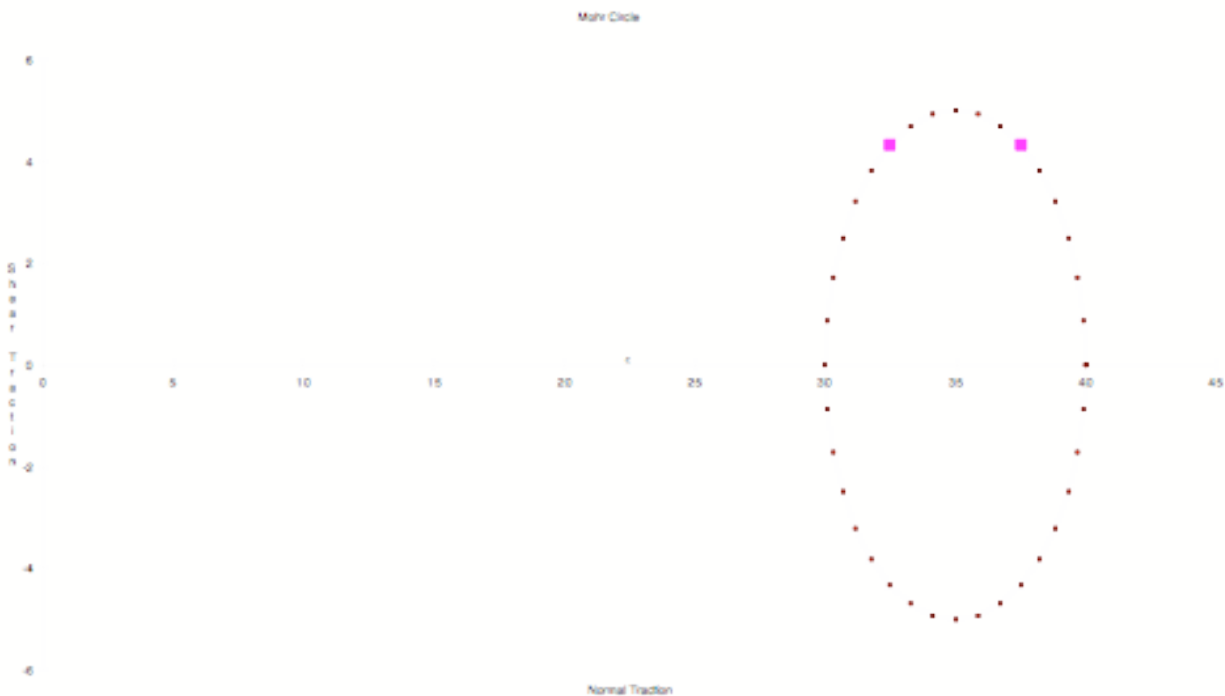
4a (5 pts) Use Excel to find the shear stress and normal stress on planes with normals oriented at 30° and 60° from the X_3 -axis.

<i>theta (deg):</i>	30	60
<i>sigmaN MPa:</i>	37.5	32.5
<i>sigmaS MPa:</i>	4.3	4.3

-2 if switched σ_n

-1 if negative σ_3

4b (5 pts) Get a printout of the Mohr circle and show where the stress components acting on these two planes plot on the circle.



-3 if points have negative shear traction

4c (5 pts) If these two planes represent potential fault surfaces, which one is more likely to fail?

HINT: Assume faulting is a frictional process driven by shear stress and resisted by compressive stress.

The plane oriented at $\theta = 60$ degrees, because there is less normal stress but the same amount of shear stress. It would cross over the coulomb failure envelope sooner than the one on the other side of the circle.

-2 if don't say why