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Effects of Bubbles on the Hydraulic Conductivity of Porous Materials – Theoretical Results

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Abstract. In a porous material, both the pressure drop across a bubble and its speed are nonlinear functions of the fluid velocity. Nonlinear dynamics of bubbles in turn affect the macroscopic hydraulic conductivity, and thus the fluid velocity. We treat a porous medium as a network of tubes and combine critical path analysis with pore-scale results to predict the effects of bubble dynamics on the macroscopic hydraulic conductivity and bubble density. Critical path analysis uses percolation theory to find the dominant (approximately) one-dimensional flow paths. We find that in steady state, along percolating pathways, bubble density decreases with increasing fluid velocity, and bubble density is thus smallest in the smallest (critical) tubes. We find that the hydraulic conductivity increases monotonically with increasing capillary number up to Ca $\sim 10^{-2}$, but may decrease for larger capillary numbers due to the relative decrease of bubble density in the critical pores. We also identify processes that can provide a positive feedback between bubble density and fluid flow along the critical paths. The feedback amplifies statistical fluctuations in the density of bubbles, producing fluctuations in the hydraulic conductivity.

Key words: bubbles, critical path analysis, percolation theory, hydraulic conductivity.

1. Introduction

The motion of bubbles in tubes is frequently studied for its value in understanding pore-scale multi-phase flow processes (Olbricht, 1996). While such model geometries are obviously gross oversimplifications of true pore geometries, the effects of bounding walls, viscosity ratio, volume fraction, bubble volume, and interfacial tension can all be considered. The model problem may thus provide physical insight into pore-scale hydrodynamics.

The presence of a bubble in a pore affects the flow in that pore, and consequently global properties such as a macroscopic hydraulic conductivity. Of course the flow in a given pore is ultimately given in terms of the macroscopic hydraulic conductivity. When bubbles are present, the flow is not of simple Darcian form. In detail, because both the pressure drop across a bubble and the speed at which the bubble moves are nonlinear functions of the fluid flux, Darcy's law is not satisfied, even at the pore-scale. The goal of the present study is to use pore-scale results to deduce properties of the porous material, in particular the macroscopic hydraulic conductivity and the density distribution of bubbles.

One approach to calculate the linear (hydraulic) conductivity of disordered networks with a wide spread in local linear conductances is to use percolation theory (Kirkpatrick, 1973a, b; Seager and Pike, 1974; Pollak, 1985; Bernabe and Bruderer, 1998; Hunt, 2001). A general nonlinear treatment of transport using percolation theory is not universally agreed on (Pollak and Riess, 1976; Shklovskii, 1976; Aladashvili and Adamia, 2002). Nevertheless, as we argue here, percolation theory can still be applied when the local conductances are nonlinear so long as neither the form of the nonlinearity nor the choice of flow paths depend on the flow.

Previous analytical and numerical studies (e.g. Ratulowski and Chang, 1989) of bubbles in porous materials have considered the effects of bubbles on the macroscopic hydraulic conductivity for ordered porous materials with a uniform (or nearly uniform) pore size. In the present work we apply percolation theory in order to derive analogous analytical results for a medium with a much wider range of tube sizes. An important aspect of our derivation is that the density of bubbles is no longer required to be uniform, rather it is determined dynamically. While our model is conceptually similar to 'droplet train' models (e.g. Rossen, 1990; Foulser *et al.*, 1991; Babchin and Yuan, 1997) we explicitly account for the nonlinear dynamics of the bubbles.

The structure of the calculations depends on both the pore-scale dynamics and the application of percolation theory to find the macroscopic response. The particular application of percolation theory depends on the model chosen. Thus, in this paper we first describe the motion of a single bubble in a single tube (Section 2) and introduce a model for the pore space topology (Section 3). Then we introduce the relevant concepts of percolation theory, of which the most important are that certain (approximately one-dimensional, 1D) pathways dominate the flow, and that the properties of flow on these pathways are dominated by particular pores (Section 4). Given the assumption of a 1D pathway, it is possible to derive a relationship between the bubble density and tube size (Section 5). In Section 6 we use the bubble density, the local nonlinear dynamics, and the results from percolation theory to obtain an estimate of the effects of bubbles on the macroscopic hydraulic conductivity. Finally, in Section 7 we acknowledge that steady-state conditions do not always exist in real flows, and discuss the possible consequences for bubble dynamics.

2. Single Bubble Dynamics

In order to develop a model for the motion of a bubbly fluid through a porous material, we represent a possibly complex pore geometry (Figure 1(a)) as a network of cylindrical tubes (Figure 1(b)). This geometric approximation is made in order to simplify the analysis. It also allows us to use well-established pore-scale results. While the geometry shown in Figure 1(b) may not resemble typical porous

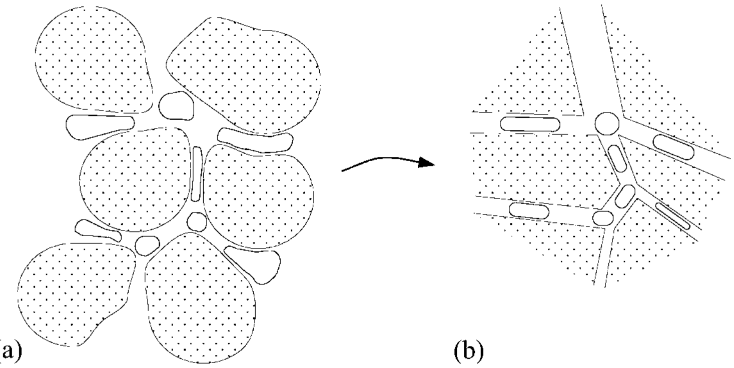

Figure 1. (a) Schematic two-dimensional view of non-cylindrical bubbles in non-cylindrical pore space. (b) An idealized version of (1a) commensurate with cylindrical geometry, represented as a network of tubes.

materials, for example, soils, we will argue in Section 4 that the macroscopic properties of the flow (in particular the hydraulic conductivity) are dominated by processes in the smaller pores through which fluid is forced to flow. For such pores our model problem may be a reasonable approximation that captures the essential physics of flow, at least for the case in which the gas phase exists as discrete bubbles.

Bretherton (1961) found that an infinitely long bubble moves at a speed u which exceeds the fluid velocity v by an amount proportional to the bubble speed, with the proportionality constant defined to be w. The relationship between u and v is therefore

$$u = \frac{v}{1-w}. \tag{1}$$

w depends on the capillary number, Ca. To leading order,

$$w = 1.29(3\mathrm{Ca})^{2/3}, \tag{2}$$

where Ca is defined by

$$\mathrm{Ca} = \frac{\mu u}{\sigma}. \tag{3}$$

Here μ is the dynamic viscosity of the suspending fluid, and σ is the interfacial tension between the bubble and the suspending fluid. Equation (2) is an excellent approximation up to $\mathrm{Ca} \sim 10^{-2}$. Although Bretherton's (1961) analysis considered infinitely long bubbles, Equation (2) applies for bubbles with volumes as small as $0.95(4\pi/3)r^3$, where r is the tube radius (Olbricht, 1996).

Because Ca depends on u, Equation (1) represents an implicit equation for w. A solution for w in terms of the fluid velocity, v, accurate over the entire range of capillary numbers investigated, is given in Stark and Manga (2000). Here, however, we solve Equation (1) numerically whenever a value is required.

The pressure drop, ΔP_{b}, across a bubble within a tube of radius r is

$$\Delta P_{\mathrm{b}} = 3.58\frac{\sigma}{r}(3\mathrm{Ca})^{2/3}. \tag{4}$$

ΔP_{b} can be rewritten in terms of w and v by using Equation (3) for Ca, and Equation (1) for u,

$$\Delta P_{\mathrm{b}} = \frac{7.45(\sigma/r)(\mu v/\sigma)^{2/3}}{[1-w]^{2/3}} \equiv \frac{7.45(\sigma/r)(\mathrm{Ca}^*)^{2/3}}{[1-w]^{2/3}}, \tag{5}$$

where Ca* is defined based on the fluid speed v rather than the bubble speed u.

The dynamics described by Equations (2) and (4) apply for surfactant-free discrete bubbles and if non-hydrodynamic effects can be ignored (reasonable for length scales greater than 10 nm). Many practical applications involving bubbles in porous materials use foams stabilized with surfactants, however (e.g. Hazlett and Furr, 2000). Although effects of surfactants on bubble motion have been studied (e.g. Ratulowski and Chang, 1990; He *et al.*, 1991; Borhan and Mao, 1992), we nevertheless choose to neglect such effects in the interests of clarity and simplicity.

One important final result is that in a chain of identical bubbles, Ratulowski and Chang (1989) noted that each bubble behaves independently. Thus, extension to treat a large number of bubbles is possible by multiplying the effects of one bubble by the number of bubbles, N.

In the rest of this paper we consider the implications of bubble dynamics. In particular we will see that for steady-state flow along 1D paths, the conservation of bubbles, and the conservation of fluid, have some interesting consequences for the variation of bubble density. The relevance of 1D flow paths is understood within the concept of percolation theory. Percolation theory has a second important implication that the dynamics of bubbles in a critical tube (of a particular radius) dominate the hydraulic properties of the entire network. In order to introduce percolation theory, however, we must first briefly define the model pore space.

3. Pore Space Topology

Some justification for fractal pore space descriptions exists (Turcotte, 1986; Sahimi, 1995). We thus consider a tube network with some fractal characteristics, and use a basic result for the porosity (Rieu and Sposito, 1991) as a test. The individual tubes are right circular cylinders with radius r, length Cr, and volume, $V = C\pi r^3$, with C large. The probability to measure a tube radius within dr of r is $W(r)\,\mathrm{d}r \propto r^{-1-D}$. In detail,

$$W(r)\,\mathrm{d}r = \frac{3-D}{r_{\mathrm{m}}^{3-D}C\pi}r^{-1-D}\,\mathrm{d}r. \tag{6}$$

This power law is compatible with a (random) fractal (of dimensionality D) arrangement of tubes with radii $r_0 \leqslant r \leqslant r_{\mathrm{m}}$. r_0 and r_{m} are the radii which bound the validity of a fractal description. The power $-1-D$ is used rather than $-D$ because, in contrast to Rieu and Sposito (1991), we are not modeling a discrete fractal, but a continuous distribution (Hunt, 2001). The proportionality constant, however, is chosen so as to produce the known result for the porosity, ϕ (Rieu and Sposito, 1991). The porosity is found from Equation (6) by integrating over all allowed r the product of $VW(r)$,

$$\int_{r_0}^{r_{\mathrm{m}}} \frac{3-D}{C\pi r_{\mathrm{m}}^{3-D}} r^{-1-D} C\pi r^3 \, \mathrm{d}r = 1 - \left(\frac{r_0}{r_{\mathrm{m}}}\right)^{3-D} = \phi. \tag{7}$$

Though the value of the permeability itself will depend on pore space topology and geometry (Katz and Thompson, 1986; Friedman and Seaton, 1998; Hunt, 2001), the structure of the result for the change in permeability due to bubbles will not depend sensitively on the choice of representation of the pore space.

4. Percolation Theory and Critical Path Analysis

One of the most accurate means to calculate transport coefficients, including the electrical and hydraulic conductivities, in disordered media (Kirkpatrick, 1973a, b; Seager and Pike, 1974; Bernabe and Bruderer, 1998; Hunt, 2001), is to apply a version of percolation theory called critical path analysis (Ambegaokar *et al.*, 1971; Shklovskii and Efros, 1971; Pollak, 1972; Katz and Thompson, 1986). Critical path analysis, explained here by an analogy to the electrical resistivity problem, exploits the following simple relationship. If two resistances differing by many orders of magnitude are configured in parallel, virtually all the current flows through the smaller resistor, and the effective resistance is equal to the smaller resistance value. If two such resistances are configured in series, virtually the entire potential drop occurs across the largest resistor, and the effective resistance is equal to the larger resistance value. By use of percolation theory it is possible to select that resistance value, out of a wide range of resistance values, that governs the current. Specifically, percolation theory can be used to find the largest resistance value, R_{c} (called the critical resistance), on the path that has the smallest possible total resistance on a transit through the system. In keeping with the above simplified discussion, other paths through the system have a much higher resistance, and may be neglected. But the most important potential drops occur over the largest resistors on the critical path. The complete treatment is more complex (e.g. Friedman and Pollak, 1981; Hunt, 2001; Skal, 2001), and the hydraulic conductivity involves additional length scales. But in many applications, knowing the critical value of the hydraulic conductance, including, if necessary, how it changes with changing conditions, has proved sufficient (see review by Hunt, 2001), and will be assumed sufficient for the present application as well.

The particular application of percolation theory depends on whether or not the pore space is represented as a regular network. On a regular network, with bonds describing resistances of varying strengths, the quantile of the distribution of resistances defining R_c is given by the bond percolation fraction, p_c. When the pore space is treated as a fractal, it has proven advantageous to apply continuum percolation (Hunt, 2001; Hunt and Gee, 2001). The reason for this is that fractal geometries require pores at all length scales to have the same aspect ratio, and inclusion of, say, two orders of magnitude of pore sizes then requires a distribution of pore lengths over two orders of magnitude. Such a wide distribution of pore lengths is difficult to constrain on a regular lattice (with translationally invariant coordination number).

When continuum percolation is applied, it is necessary to consider the total volume occupied by pores (Hunt, 2001). An example can be the network of tubes considered here. When a portion of the tube network, including tubes with the largest radii, is emplaced spatially at random (with no spatial correlations), a particular volume fraction is associated with the tubes. If enough tubes are included so that the set of tubes interconnects to infinite size, the critical value, α_c, of the volume fraction has been reached, and the tube volumes are said to percolate. This critical volume fraction is not known a priori for arbitrary geometries and topologies of connection. Nevertheless a typical value of $\alpha_c \approx 0.15$ can be used with some justification (Stauffer, 1975) although higher values are also possible (Balberg, 1985; Moldrup *et al.*, 2001).

What is the critical radius for percolation when the tube size distribution is consistent with a fractal, as described in the previous section? To find the critical radius, r_c, it is necessary to integrate over the distribution of r-values from the critical value to the largest allowed in the distribution, and set that pore space volume equal to the critical volume fraction, α_c,

$$\frac{3-D}{r_m^{3-D}} \int_{r_c}^{r_m} r^{2-D}\, dr = \alpha_c. \tag{8}$$

Solving Equation (8) for r_c in terms of α_c, we obtain

$$r_c = r_m(1-\alpha_c)^{1/(3-D)}. \tag{9}$$

This is the smallest radius of any tube that the fluid must flow through to get from one side of a system to the other. It has been conjectured that for fractal soils, the fractal dimensionalities of the solid volume and the pore space are very similar (Rieu and Sposito, 1991). For soil particle-size distributions, typical values of D are 2.8 ± 0.1 (Wu *et al.*, 1993; Bittelli *et al.*, 1999) leading to values of r_c which are between 2 and 5 times smaller than r_m, if $\alpha_c = 0.15$ is used. Because such fractal descriptions may be valid over orders of magnitude of pore sizes (Bittelli *et al.*, 1991; Perrier *et al.*, 1996; Hunt and Gee, 2001) r_c can often be more than an order of magnitude larger than r_0.

The value of the critical radius for percolation is not changed by considering nonlinearities in flow due to bubbles under the following conditions, both of which should apply for the problem being considered:

1. there is a unique relationship between the individual hydraulic conductance and the tube radius, that is, it is not possible to get two different conductivities with one tube radius, nor is it possible to get the same conductance from two different tube radii,
2. the tube conductance is an increasing function of tube radius.

The first condition holds whenever the initial bubble distribution is constrained to be homogeneous in space. The reason is that it is, in this case, always possible to use transformation of variables in the integration to return to the same integral condition on r. Nevertheless, the condition is perhaps most easily understood by considering its violation through, for example, an inhomogeneous bubble distribution. If, at low Ca, two tubes with very slightly differing r values have greatly differing bubble contents, the tube with the larger r will have greater flow only if it has fewer bubbles. Under these circumstances, both the number of bubbles and the tube dimensions can be considered as independent random variables, and it is necessary to find the critical value of the hydraulic conductance $g(r, \phi_b)$. Such problems coupling different random variables tend, in 3D, to lead to larger values of a critical conductance, but smaller values of the smallest r value encountered along a percolation path. An analogous effect is considered in detail in Hunt (2001) where for the electrical conductivity both the radius and the ionic strength are treated as random variables. If the first condition mentioned above holds, but the latter condition is violated, it is necessary only to change the order of integration from the original smallest r required on the path to r_m, to a final range from r_0 to the largest r required on the path.

The calculated value of r_c is useful to define a critical value of the hydraulic conductance,

$$g_c^0 = \frac{\pi \rho r_c^4}{8\mu l_c} = \frac{\pi \rho r_c^3}{8\mu C}, \tag{10}$$

where the superscript 0 indicates that this is the value of g_c in the absence of bubbles, and subscripts c indicate values for the critical pore. An added complication is the necessity, in steady state, to guarantee a time-independent flux of bubbles as well as a time-independent fluid flux. This problem is dealt with next.

5. Spatial Distribution of Bubbles

We consider the case for which the bubble distribution has no random component. Associated with the bubbles is a fraction of the total pore space, which we call the bubble pore space, with bubble porosity, ϕ_b, normalized to the porosity of the medium so that $0 \leqslant \phi_b < 1$. The number, N, of bubbles in a given tube of volume V depends on the bubble volume, $V_b \equiv (4\pi/3)r_b^3$, as well as on ϕ_b,

$$N = \phi_b \frac{V}{V_b}. \tag{11}$$

Substituting the tube volume $V = Cr(\pi r^2)$ for a tube of radius r and length Cr (constant aspect ratio, C) gives

$$N = \frac{\pi C r^3}{V_b}\phi_b = \frac{3}{4}C\left(\frac{r}{r_b}\right)^3\phi_b. \tag{12}$$

The number of bubbles per unit length in such a tube is

$$\frac{\pi r^2}{V_b}\phi_b. \tag{13}$$

The actual length of a bubble is

$$d = V_b/\pi r^2, \tag{14}$$

so that the total length taken up by bubbles is Nd,

$$Nd = \frac{\pi C r^3}{V_b}\frac{V_b}{\pi r^2}\phi_b = Cr\phi_b. \tag{15}$$

Now it is possible to write for the effective length (bubble-free portion) of the tube,

$$l^* = Cr[1 - \phi_b]. \tag{16}$$

By using percolation theory to find the macroscopic hydraulic conductivity we can make the first order approximation that flow occurs along one-dimensional paths, and that off such paths there is comparatively little flow. As a consequence, it is possible to derive some simple relationships between bubble density and fluid flow velocities for tubes along the percolation path.

Consider a tube of radius r on the 'percolation path.' The mean distance between bubbles in such a tube is

$$d_s = \frac{V_b}{\pi r^2\phi_b}. \tag{17}$$

In time Δt the mean number of bubbles escaping a tube of radius, r, is

$$\frac{u\Delta t}{V_b/(\pi r^2)}\phi_b. \tag{18}$$

This number must be identical for all tubes on the percolation path (assuming, in a lowest order approximation, 1D flow along this path), and therefore

$$u\pi r^2\phi_b = \frac{v}{1-w}\pi r^2\phi_b = \text{ constant}. \tag{19}$$

In addition, the flux of fluid volume, Q, must be identical everywhere along this path (under steady-state conditions). Using the result that $v = Q/\pi r^2$ yields

$$\phi_b \propto (1 - w). \tag{20}$$

Thus on the percolation path (with the largest fluid speeds, and hence, the largest bubble fluxes) bubble density in steady-state flow is not uniform, but depends on the tube radius, r, that is, $\phi_b = \phi_b(r)$.

Under steady-state conditions the mean concentration of bubbles along the percolation paths is the same as the injection concentration, $\overline{\phi}_b$. Rewrite Equation (20) as

$$\phi_b(r) = \phi_0(1 - w(r)) \tag{21}$$

with ϕ_0 a constant. The integral of ϕ_b over all tubes found along the critical path for percolation is equal to the mean value, $\overline{\phi}_b$, of the bubble porosity by assumption, so that it is possible to solve for ϕ_0,

$$\phi_0 = \frac{\overline{\phi}_b \int_{r_c}^{r_m} W(r)\,\mathrm{d}r}{\int_{r_c}^{r_m} (1 - w(r))W(r)\,\mathrm{d}r}. \tag{22}$$

The upper limit of integration, r_m, is the largest tube radius in the system, while the lower limit, r_c, is the smallest tube encountered along the percolation path. Using Equation (22) for ϕ_0, we obtain

$$\phi_b(r) = \frac{\overline{\phi}_b[1 - w(r)] \int_{r_c}^{r_m} W(r)\,\mathrm{d}r}{\int_{r_c}^{r_m} (1 - w)W(r)\,\mathrm{d}r} \equiv \frac{\overline{\phi}_b(1 - w(r))}{1 - \overline{w}}, \tag{23}$$

where $\overline{w}$ is the average value of w along the percolation path. For a system with a wide distribution of pore sizes, such as a fractal distribution with $3 - D \sim 0.2$, $w_c/\overline{w} > 1$ so that

$$\phi_b(r_c) \approx \overline{\phi}_b(1 - w(r_c)). \tag{24}$$

6. Application of Percolation Theory to Flow with Nonlinear Bubble Dynamics

The fluid flux in a tube of radius r is given in terms of the pressure drop in the tube, less the pressure drops across the bubbles. Because the bubbles are identical, we need to multiply ΔP_b in Equation (5) by N from Equation (12), to find the total pressure drop across the bubbles, and subtract this value from the total pressure drop in the tube. The remaining pressure drop, which acts on the fluid, thus acts only on the length of tube which is bubble-free. So the length, l, which appears in Equation (10) must be replaced by l^* from Equation (16). Now it is possible to write a self-consistent equation for Q for any of the tubes on the 'percolation path,'

$$Q = \frac{\pi \rho r^4}{8\mu C r[1 - \phi_b(r)]} \times \left[1 - \frac{V}{V_b}\overline{\phi}_b \frac{1}{\Delta P} \frac{7.54\sigma}{r} \frac{[1 - w(r)]^{1/3}}{[1 - \overline{w}]^{2/3}} (\mathrm{Ca}^*)^{2/3}\right] \Delta P, \tag{25}$$

where V is the volume of the tube under consideration, and ϕ_b is given in Equation (23). Using the definition (Equation (10)) of the hydraulic conductance in the absence of bubbles, Equation (25) can be rewritten,

$$Q = \frac{g^0}{1-\phi_b(r)}\left[1 - \frac{V}{V_b}\overline{\phi}_b \frac{1}{\Delta P}\frac{7.54\sigma}{r}\frac{[1-w(r)]^{1/3}}{[1-\overline{w}]^{2/3}}(\mathrm{Ca}^*)^{2/3}\right]\Delta P. \tag{26}$$

In order to characterize our results, we use the procedure of Stark and Manga (2000), and determine the effect of the presence of bubbles on the hydraulic conductivity by comparing the conductivity with bubbles to its value without bubbles. This procedure amounts to setting the values of ΔP across the critical conductance (and hence, critical tube) in the presence and in the absence of bubbles equal, and solving for the ratio of the resulting discharges, Q. We thus obtain

$$\begin{aligned}&\frac{8\mu C[1-(1-\phi_b(r_c)]Q_c}{\pi\rho r_c^3} + \frac{V_c\overline{\phi}_b\sigma}{V_b r_c}\left[7.54(\mathrm{Ca}_c^*)^{2/3}\left[\frac{1-w(r_c)}{(1-\overline{w})^2}\right]^{1/3}\right]\\ &\quad = \frac{8\mu C Q_c^0}{\pi\rho r_c^3}.\end{aligned} \tag{27}$$

Equation (27) can be solved for the ratio of Q_c/Q_c^0 to yield

$$\begin{aligned}\frac{g_c}{g_c^0} = \Bigg\{&1 - \left[\frac{1-w(r_c)}{1-\overline{w}}\right]\overline{\phi}_b + \\ &\frac{\pi r_c^3\overline{\phi}_b}{V_b}\left[0.931(\mathrm{Ca}_c^*)^{-1/3}\left[\frac{1-w(r_c)}{(1-\overline{w})^2}\right]^{1/3}\right]\Bigg\}^{-1}.\end{aligned} \tag{28}$$

Equation (28) provides an expression for effects of bubbles on the hydraulic conductivity, when combined with information about the pore geometry, in particular r_c from Equation (9) and $\overline{w}$ from Equation (23). g_c/g_c^0 is equivalent to the liquid phase relative permeability; a relative permeability for the gas phase is not meaningful as the bubbles are not continuous.

One last point needs to be addressed. It is tacitly assumed above that by not changing the value of the externally applied pressure, the value of ΔP that appears in Equations (25) and (26) is not changed. This is justified within the basic approximation of critical path analysis that all (or nearly all) the pressure drop in the network occurs across critical valued resistances along the percolation path. Thus it is necessary only to find out how far apart such critical resistance values occur, a distance we call l_c. Given a value for l_c, it is possible to write

$$\Delta P_c = \Delta P\left(\frac{l_c}{z}\right), \tag{29}$$

where z is the total width of the system (over which the pressure difference ΔP is applied), and ΔP_c is the pressure difference applied across a critical tube (with r_c as calculated in the absence of bubbles). Since, however, there has been no

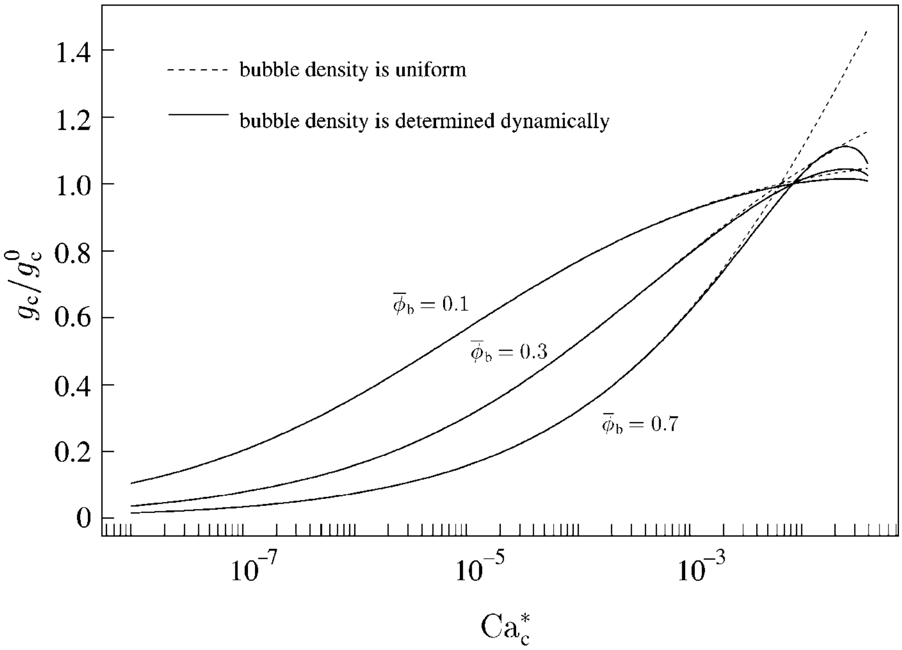

Figure 2. Modification of the system hydraulic conductivity as a function of Ca referenced to the fluid flow velocity, and as a function of bubble density.

change in r_c due to the introduction of the bubbles (only in Q_c), it is reasonable to assume that l_c does not change either. It is therefore unnecessary to know what ΔP is because the same fraction of the macroscopic pressure drop is still recorded across the critical resistances. The previous argument implies that the change in conductance of the critical tube (in the form of a ratio) is likely to be the same as the result for the entire network. In reality, however, the presence of bubbles reduces the conductance of smaller tubes less (with larger Ca, see Figure 2) than that of larger tubes, thus changing ΔP across the critical tubes slightly upon introduction of bubbles.

The results of Equation (28) for various bubble concentrations as a function of Ca* are shown in Figure 2. Our numerical solution uses the approximation, $w(r_c)/\overline{w} \gg 1$. For Ca* less than 10^{-2}, the hydraulic conductivity is depressed by the presence of bubbles. The effect is most pronounced at small values of the capillary number and large bubble concentrations, a well-known result for foams (e.g. Ratulowski and Chang, 1989; Patzek, 2001). For example, for bubble concentrations $\phi_b = 0.7$ and $Ca^* \sim 10^{-7}$, the reduction is two orders of magnitude. At values of $Ca^* > 10^{-2}$ bubbles enhance the hydraulic conductivity. This enhancement is strongest at the largest bubble concentrations. The form of the curves for g_c/g_c^0 shown in Figure 2 is similar to those obtained numerically by Stark and Manga (2000), especially for $Ca^* < 10^{-3}$.

7. Unsteady Flow and Flow Instabilities

Up to this point, our analysis assumes that steady-state conditions are obeyed. But an interesting consequence of the present results is that they support the existence of flow fluctuations that are amplified by the nonlinear dynamics. To see this consider again Figure 2 in the regime of low Ca*. Assume a drastically simplified model of the percolation path, with alternating tubes of radii r_c and $5r_c$. The tubes with radius $5r_c$ have 125 times the conductance, so the potential drops across them are only 1/125 times as large, in the mean, in order to produce the same flow, Q. The tubes with $r = r_c$, on the other hand, have a 25 times larger value of a mean v, on account of their smaller r-values, and thus have larger values of the capillary number (both Ca and Ca*).

Consider now the possibility that a fluctuation in bubble numbers occurs such that the number of bubbles in the tubes with $r = r_c$ drops. Their conductances are increased and the net flow increases as well, whether or not the bubble number in the larger tubes increases. Because, to first approximation, 125/126 of the pressure drop occurs across the critical tube, the larger tubes can increase their flow (and pressure drop) significantly without any significant change in the pressure drop across the critical tube. Thus v increases for both tubes, and by the same fractional increment. Such a proportional increase in v, however, causes a disproportionate increase in bubble velocity, u, because the critical tubes have a larger v, and the rate at which u increases with v increases with increasing u. Thus the bubble velocity is increased by a larger fraction in the critical tubes than in the larger tubes. Such a disproportionate increase tends to reduce the number of bubbles in the critical tubes even further, and this positive feedback leads to amplifications of flow fluctuations.

By an analogous argument, an increase in bubble concentration in the critical tubes (at low Ca) leads to a further increase, with the result that these tubes can become blocked, a process we neglect in our analysis. In response, the fluid may find another path (with a larger value of the maximum resistance) leading to a fluctuation in flow paths, and hence the hydraulic conductivity. When bubbles are first injected into a porous medium, each blocking of a critical pore may be accompanied by the trapping of air in pores. If so, we expect to see a trend of diminishing water content and associated reduction in hydraulic conductivity along with the flow fluctuations, such as those described in Faybishenko (1995, 1999, 2000), which include both laboratory and field measurements.

Stark and Manga (2000) simulated numerically nonlinear bubble flow in the same model considered here, but with less extreme geometry (ratio of maximum to minimum pore size is of a factor 3), and observed that, “At low Ca, the presence of one or more bubbles on a high-flow pathway can significantly reduce the flow through that pathway. Other bubbles are thus more likely to leave that high-flow pathway, and the presence of nodes enables bubbles to avoid congested pathways." Indeed, in Figure 2(a) of Stark and Manga (2000) there are significant mean velocity fluctuations that are much larger than those due to variations in bubble content alone (compare with the magnitude of the bubble density fluctuations in their Figure 2(b)).

For cases in which the critical tubes have large Ca values (above 10^{-2}) a reduction in bubble density in the critical tubes leads to a diminution of flow, because the flow here increases with bubble density. Consequently a greater reduction in *u* occurs in the critical tubes (with the larger Ca) and bubble concentrations increase in the critical tubes. Thus, when the critical tubes are in the large Ca region, fluctuations in bubble density are attenuated. This argument does not depend on the value of Ca^* in the larger tubes.

8. Conclusions

We have treated nonlinear pore-scale dynamics of bubble and fluid flow in porous media using a network of tubes with some characteristics of a fractal. The calculations of the macroscopic system response utilized critical path analysis, defined in terms of percolation theory. In this analysis the predominant fluid flow and bubble transport occur along quasi-one-dimensional paths of interconnected tubes. The smallest tubes through which the bubbly fluid must flow dominate both the flow and bubble transport properties. Conservation of bubbles in steady state requires a systematic variation of bubble density with tube radius on these dominant paths: larger tubes have smaller capillary numbers and higher bubble densities.

We find that K is reduced by the presence of bubbles when the critical tube for flow is in a low capillary number regime ($Ca^* < 10^{-3}$), but is enhanced for larger values of Ca. In both cases, the effects increase with increasing bubble density. The dynamic redistribution of bubbles has little effect on the system-wide hydraulic conductivity, except at $Ca^* > 10^{-3}$, where it reduces the enhancement of the conductivity resulting from the presence of bubbles. Although the preceding results should hold for steady-state flow, we find that the nonlinearities associated with bubble transport amplify flow fluctuations due to fluctuations in bubble density when Ca^* in the critical tube is less than 10^{-3}. At higher values of Ca^*, the enhancement of the hydraulic conductivity from the presence of bubbles suppresses flow variations due to fluctuations in bubble density.

We must reemphasize the significant geometric simplifications in the model problem. Also, the critical path analysis is essentially one-dimensional. Moreover, approximating the dispersed gas phase as discrete bubbles will not describe the topology of the gas phase in the general case (Avraam and Payatakes, 1995), although it may describe situations such as foam mobilization (Rossen *et al.*, 1994). Still, the theoretical results developed here provide a framework for interpreting experimental results and numerical simulations. Differences between measurements and our results should in principle provide insight into those processes not included in the model.

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