

# Boundary Integral Methods for Viscous Free-Boundary Problems: Deformation of Single and Multiple Fluid-Fluid Interfaces

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## ABSTRACT

This article reviews the application of boundary integral methods to low Reynolds number free-boundary flows. The basic equations are developed for three prototypical problems: (i) a drop in an unbounded fluid, (ii) a rigid sphere translating towards a fluid-fluid interface, and (iii) a drop moving through an interface. The interfacial velocity field is expressed as a Fredholm integral equation of the second kind, where the integration domain is the deforming interface. This velocity coupled with the kinematic condition determines the interface evolution. A brief discussion is given of the numerical treatment of the equations, and an illustration is given of time-dependent deformation for several two-interface problems. An extensive literature review is provided.

## 1. INTRODUCTION

Low Reynolds number flows are characterized by small velocities, small length scales and/or high fluid viscosities. In this limit the inertia terms in the Navier-Stokes equations are neglected; the fluid motion is governed by the linear (quasi-steady) Stokes equations [16, 19]. Integral equation formulations are a natural choice for certain classes of Stokes flow problems [77]. The methods may be applied to flows involving only rigid boundaries (*e.g.*, a suspension of rigid particles) and flows involving at least one fluid-fluid interface, hereafter referred to as *free-boundary* problems.<sup>1</sup> The former subject has been discussed and developed by Kim and coworkers [26]. This paper presents a brief survey of the application

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<sup>1</sup>Boundary integral methods have also been used to study high Reynolds number free-boundary problems. This is a separate subject and the interested reader can refer to [4, 38, 48].

of boundary integral methods to free-boundary problems for flows at low Reynolds numbers.

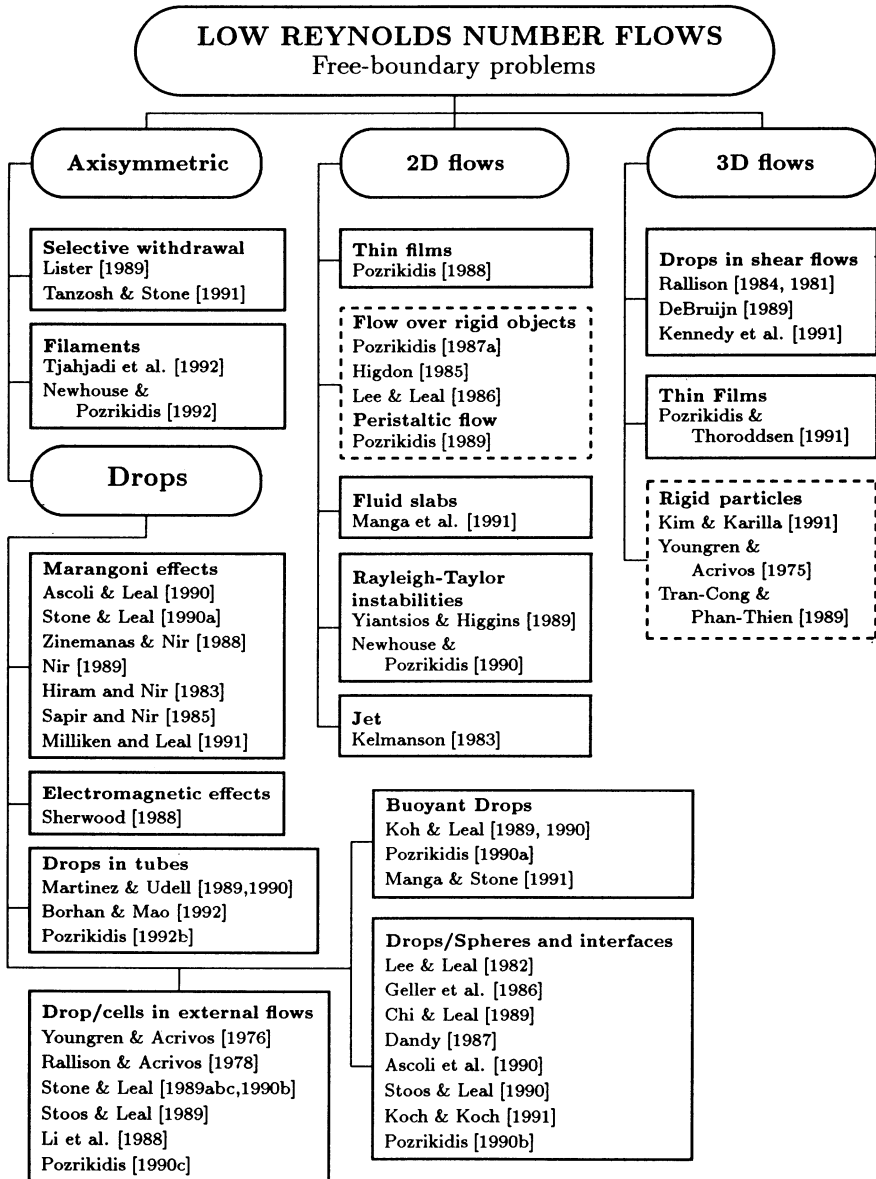
The dynamics of interface deformation in low Reynolds number flows is of interest in a wide variety of fields including chemical and petroleum engineering, solid-earth geophysics, hydrology, and biology. Typical applications span an immense range of length scales from microns to hundreds of kilometers: biological studies of cell deformation; chemical engineering studies of coalescence, flotation, coating flows and the dynamics of thin films; and geophysical studies of mantle plumes, lithospheric slabs and magma chambers.

The principal difficulty with solving free-boundary problems is that the position of the interface is unknown *a priori* and must be determined as part of the solution. Thus, the problem of determining the time-dependent interface shape is inherently non-linear.

The boundary integral method relates velocities at points within the fluid to the velocity and stress on the bounding surfaces. It is an ideal method for studying free-boundary problems. Advantages of the technique include the reduction of problem dimensionality, the direct calculation of the interfacial velocity, the ability to track large surface deformations, and the potential for easily incorporating interfacial tension as well as other surface effects (*e.g.*, electric field-induced stresses).

The boundary integral formulation for Stokes flows was theoretically described by Ladyzhenskaya within the framework of hydrodynamic potentials [30]. This integral equation method was developed and implemented numerically by Youngren & Acrivos [80] in a study of the translation of arbitrarily-shaped rigid particles. Shortly thereafter, the method was extended to the study of deformation of fluid-fluid interfaces: the deformation of bubbles and drops in extensional flows [63, 81] and the motion of a rigid sphere moving normal to a deformable interface [31]. In recent years the number of applications has increased enormously.

Applications of boundary integral methods have ranged from the classical study of a rising drop in an otherwise quiescent fluid [28, 29, 56], to more complex situations such as drop breakup in extensional flows, the deformation of a blood cell [33, 57], and the deformation of small drops in electric and magnetic fields [66]. The integral equation approach has also been applied to study the growth of two-dimensional Rayleigh-Taylor instabilities [45, 79] and the selective withdrawal of fluid from a stratified layer [36]. Although most work has been concerned with axisymmetric or two-dimensional interface deformations, which only require numerical treatment of line integrals, several studies have considered the more difficult case of three-dimensional surface distortion [13, 25, 60, 61]. Recent studies have combined boundary integral methods with lubrication theory



**Figure 1:** Summary of free-boundary problems studied numerically using the boundary integral formulation for low Reynolds numbers flows. Selected studies involving flow over rigid surfaces are included (dashed boxes).

| Surfaces     |        | Investigators  | External Forcing                | Buoyancy                        | Interfacial Tension             | Other                           | Geometry                                | Comment/Keywords  |
|--------------|--------|--|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---|---|
| Fluid        | Rigid  |  |                                 |                                 |                                 |                                 |   |   |
| drop         | —      | [81], [63], [67], [69]<br>[61], [13], [25]<br>[28], [56]<br>[68]<br>[66]<br>[70], [43]<br>[82], [33] | •<br>•<br>•<br>•<br>•<br>•<br>• | •<br>•<br>•<br>•<br>•<br>•<br>• | •<br>•<br>•<br>•<br>•<br>•<br>• | •<br>•<br>•<br>•<br>•<br>•<br>• | AS<br>3-D<br>AS<br>AS<br>AS<br>AS<br>AS | Drop in extensional flow<br>Drop in simple shear flow; effective prop. emulsion<br>Stability of translating drop<br>Relaxation, capillary breakup of extended drop<br>Drop in electric/magnetic field<br>Surfactants, Convective-diffusive transport of<br>Drop w/elastic membrane, w/anisotropic surf.tens.; capsule |
| layer (thrd) | —      | [36]<br>[75]   | •<br>•                          | •<br>•                          | •<br>•                          | •<br>•                          | AS<br>AS                                | Sink near initially flat interface; singularity<br>Capillary breakup, satellite formation from fluid thread   |
| layer        | par-   | [31], [15], [72]<br>[73]   | •<br>•                          | •<br>•                          | •<br>•                          | •<br>•                          | AS<br>AS                                | Rigid particle moving toward interface<br>Particle imbedded in interface ; Contact line   |
| drop         | wall   | [3], [55]<br>[2]<br>[9]<br>[41], [42], [7]<br>[59]   | •<br>•<br>•<br>•<br>•           | •<br>•<br>•<br>•<br>•           | •<br>•<br>•<br>•<br>•           | •<br>•<br>•<br>•<br>•           | AS<br>AS<br>AS<br>AS<br>AS/P            | Buoyancy driven drop toward rigid planar wall<br>Thermocapillary migration of drop near wall<br>Growth of plume from source in wall<br>Drop in cylindrical pipe; surfactant transport<br>Periodic drop train in pipe; buoyancy driven   |
| layer        | wall   | [79], [45]<br>[53]   | •<br>•                          | •<br>•                          | •<br>•                          | •<br>•                          | 2-D/P<br>2-D/P                          | Rayleigh-Taylor in thin film over rigid wall; plumes<br>Film flowing down periodic rigid boundary   |
| (thrd)       | (pipe) | [60]<br>[46]   | •<br>•                          | •<br>•                          | •<br>•                          | •<br>•                          | 3-D<br>AS                               | Film flowing down rigid boundary with bump<br>Breakup of liquid thread inside pipe  |
| drops (2)    | —      | [71]   | •                               | •                               | •                               | •                               | AS                                      | Double emulsion drops in extensional flow   |
| drop/layer   | —      | [8]  | •                               | •                               | •                               | •                               | AS                                      | Drop moving through deformable interface  |

Table 1: (Caption on adjacent page)

to characterize the fluid motion in thin regions between two fluid layers [5, 11, 37, 78]. Many aspects related to integral equation methods for free-boundary problems are described in a recent book by Pozrikidis [58].

Figure 1 summarizes applications of integral equation methods to free-boundary problems. The studies have been classified according to whether the geometry is axisymmetric, two-dimensional, or three-dimensional. Drop problems are further subdivided according to application. Table 1 provides a more detailed summary. The problems have been classified according to the number and form of the bounding interfaces. The primary ( $\bullet$ ) and secondary ( $\circ$ ) driving forces (*e.g.*, externally-driven, interfacial-tension-driven or buoyancy-driven) responsible for the deformation are also indicated.

We proceed by considering some of the details of the boundary integral formulation. In Section 2 we provide a derivation of the basic integral equations, along with the modifications necessary for the study of more complicated problems. In Section 3, we outline the numerical techniques typically applied in the treatment of free-boundary problems and, in Section 4, present a summary of some of the research in progress in our group.

## 2. ANALYTIC FORMULATION

In this section we present the basic equations necessary for studying free-boundary problems using integral equation methods. A variety of free-boundary problems can be posed, each involving different combinations of fluid-fluid interfaces, rigid boundaries and external flows. A generic treatment covering all possibilities is rather difficult, so details of the method are discussed within the framework of several prototypical problems: (*i*) a drop in an unbounded fluid, including the presence of an external flow, §2.4; (*ii*) a rigid particle translating toward (and through) a deformable interface, §2.5; and (*iii*) a drop translating toward a deformable interface, §2.6. These free-boundary problems illustrate the basic elements necessary for treatment of, respectively, (*i*) unbounded and external flows, (*ii*) rigid boundaries, and (*iii*) multiple fluid-fluid interfaces. In principle, generalization to other geometries is straightforward.

### 2.1 Equations and boundary conditions

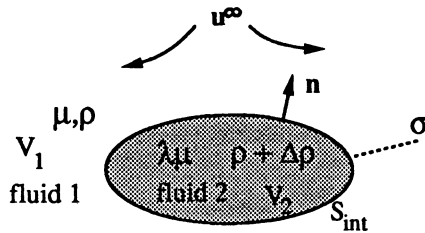
In the low Reynolds number limit, incompressible fluid motion is governed by the Stokes and continuity equations

$$\nabla \cdot \mathbf{T} = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{g} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0, \quad (1)$$

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**Table 1:** Research organized according to surface geometry – Fluid interface: open (layer, thread) or closed (drop, capsule); Rigid interface: none (—), open (rigid wall/pipe) or closed (rigid particle). The numbers reference the bibliography. Forcing:  $\bullet$  refers to primary forcing;  $\circ$  secondary forcing. Geometry: axisymmetric (AS); three-dimensional (3-D); two-dimensional (2-D) and/or periodic in one direction (P).

where  $\mathbf{u}$  is the velocity,  $p$  is the pressure,  $\mu$  and  $\rho$  are the fluid viscosity and density, respectively, and  $\mathbf{g}$  is the gravitational acceleration. Here the stress tensor  $\mathbf{T}$  is defined to include the hydrostatic body force,  $\mathbf{T} = -(p - \rho \mathbf{g} \cdot \mathbf{x}) \mathbf{I} + \mu (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ , in order to define a divergence-free field. The body force will appear explicitly in the boundary conditions if there are density contrasts across fluid-fluid interfaces. The low Reynolds number assumption requires  $\mathcal{R} = \rho u l / \mu \ll 1$  where  $u$  and  $l$  are characteristic velocity and length scales of the fluid motion.



**Figure 2:** A translating and deforming drop in an unbounded fluid.

For illustrative purposes, consider the case of a drop of viscosity  $\lambda\mu$  immersed in a second immiscible fluid of viscosity  $\mu$  with an externally-imposed velocity field  $\mathbf{u}^\infty(\mathbf{x})$  (Fig. 2). The boundary conditions at a fluid-fluid interface  $S_{int}$  require a continuous velocity and a balance between the net surface traction and interfacial tension forces. Hence,

$$\mathbf{u}_1(\mathbf{x}) \rightarrow \mathbf{u}^\infty(\mathbf{x}) \text{ as } |\mathbf{x}| \rightarrow \infty, \quad \mathbf{u}_1(\mathbf{x}) = \mathbf{u}_2(\mathbf{x}) \text{ for } \mathbf{x} = \mathbf{x}_s \in S_{int} \quad (2)$$

and the stress jump  $[\mathbf{n} \cdot \mathbf{T}]$ , accounting for a density difference across the interface, is given by (the normal  $\mathbf{n}$  points *into* fluid 1)

$$[\mathbf{n} \cdot \mathbf{T}] \equiv \mathbf{n} \cdot \mathbf{T}_1 - \mathbf{n} \cdot \mathbf{T}_2 = \sigma (\nabla_s \cdot \mathbf{n}) \mathbf{n} - \nabla_s \sigma - \Delta \rho (\mathbf{g} \cdot \mathbf{x}) \mathbf{n} \text{ for } \mathbf{x}_s \in S_{int}. \quad (3)$$

Here  $\sigma$  denotes the interfacial tension,  $\Delta \rho$  is the density difference,  $\nabla_s = (\mathbf{I} - \mathbf{n}\mathbf{n}) \cdot \nabla$  is the gradient operator tangent to the interface,  $\nabla_s \cdot \mathbf{n}$  is the mean curvature of the interface and  $\mathbf{x}_s$  denotes a point on the interface. The stress balance (3) includes both a normal stress jump, proportional to the product of interfacial tension and the local curvature, and a tangential stress jump,  $-\nabla_s \sigma$ , owing to variations in interfacial tension along the surface. These variations may result from either temperature gradients or the presence of surfactants in the fluid. In such instances, additional field equations determine the temperature profile or the distribution of surfactants along the interface [2, 70]. Other stresses may be included in the stress jump boundary condition. For example, in studies of drop deformation, Sherwood [66] examined the effect of electric or magnetic

field-induced stresses and Li, *et al.* [33] studied the role of elastic stresses generated in a thin membrane surrounding a drop.

A kinematic constraint relates changes in the interface position to the local velocity. For example, the interface evolution may be described with a Lagrangian representation  $dx_s/dt = \mathbf{u}(\mathbf{x}_s)$ .

Although time-dependence does not appear explicitly in Stokes equations, it is consistent to study time-dependent interface distortions. This *quasi-static* assumption requires that  $l^2/\nu \ll \tau$  where  $\nu$  is the kinematic viscosity, and  $\tau$  is a typical time for a change of the flow or geometry. Generally,  $\tau = \min[O(l/u^\infty), O(l\mu/\sigma)]$ , where the former is a time scale for an externally-driven flow, and the latter is the time scale of an interfacial-tension-driven motion. In most situations, the larger of the two fluid viscosities should be used for these estimates. Physically, the quasi-static approximation means that the fluid immediately adjusts to changes in the boundary location owing to rapid vorticity diffusion.

## 2.2 Green's functions

In order to derive an integral representation for Stokes flows, the fundamental singular solutions for Stokes equations are needed. These singular solutions correspond to the velocity and stress fields at a point  $\mathbf{x}$  produced by a point force  $\mathbf{F}$  located at  $\mathbf{y}$  and may be derived, for example, using Fourier transforms [26]. Denoting the singular solution by a  $\hat{\cdot}$  and solving  $\nabla \cdot \hat{\mathbf{T}} + \mathbf{F} \delta(\mathbf{x} - \mathbf{y}) = 0$  with  $\nabla \cdot \hat{\mathbf{u}} = 0$  and  $|\hat{\mathbf{u}}|, |\hat{\mathbf{T}}| \rightarrow 0$  as  $|\mathbf{x}| \rightarrow \infty$  yields

$$\hat{\mathbf{u}}(\mathbf{x}) = \frac{1}{\mu} \mathbf{J}(\mathbf{x}|\mathbf{y}) \cdot \mathbf{F} \quad \text{and} \quad \hat{\mathbf{T}}(\mathbf{x}) = \mathbf{K}(\mathbf{x}|\mathbf{y}) \cdot \mathbf{F}, \quad (4)$$

where the kernels, or Green's functions, are denoted  $\mathbf{J}$  and  $\mathbf{K}$ . The free-field Green's functions mapping a force at  $\mathbf{y}$  to the field at  $\mathbf{x}$  in an unbounded three-dimensional domain are

$$\mathbf{J}(\mathbf{x}|\mathbf{y}) = \frac{1}{8\pi} \left( \frac{\mathbf{I}}{r} + \frac{\mathbf{r}\mathbf{r}}{r^3} \right) \quad \text{and} \quad \mathbf{K}(\mathbf{x}|\mathbf{y}) = -\frac{3}{4\pi} \frac{\mathbf{r}\mathbf{r}\mathbf{r}}{r^5} \quad (\mathbf{r} = \mathbf{x} - \mathbf{y}, r = |\mathbf{r}|) \quad (5)$$

and in an unbounded two-dimensional domain are

$$\mathbf{J}(\mathbf{x}|\mathbf{y}) = \frac{1}{4\pi} \left( \mathbf{I} \log r - \frac{\mathbf{r}\mathbf{r}}{r^2} \right) \quad \text{and} \quad \mathbf{K}(\mathbf{x}|\mathbf{y}) = \frac{1}{\pi} \frac{\mathbf{r}\mathbf{r}\mathbf{r}}{r^4}. \quad (6)$$

Green functions for other geometries have also been derived. In practice, these modified Green's functions reduce the number of boundary conditions which must be imposed in boundary integral applications, thereby simplifying the problems. Examples are listed in Table 2.

**Table 2:** Selected Green's functions for Stokes flow.

| <i>Geometry</i>  | <i>Reference</i>                       |
|--|--|
| Rigid plane wall   | Blake [6]                              |
| Semi-infinite plate                                      | Hasimoto et al. [17]                   |
| Solid sphere   | Oseen (as cited in Kim & Karrila [26]) |
| Two intersecting planes                                  | Sano & Hasimoto [64]                   |
| Inside a circular cylinder                               | Liron & Shahar [35]                    |
| Solid planar wall with a hole                            | Davis [10], Miyazaki & Hasimoto [44]   |
| Between two parallel plates                              | Liron & Mochon [34]                    |
| 2-D Periodic: Symmetric with respect to a point or plane | Pozrikidis [51, 52]                    |

### 2.3 Integral representation of Stokes equations

For two solenoidal velocity fields  $(\mathbf{u}, \mathbf{T})$  and  $(\hat{\mathbf{u}}, \hat{\mathbf{T}})$  it is straightforward to derive the Reciprocal theorem (a Green's theorem), which states

$$\int_V [\mathbf{u}(\mathbf{x}) \cdot (\nabla \cdot \hat{\mathbf{T}}) - \hat{\mathbf{u}} \cdot (\nabla \cdot \mathbf{T})] dV_x = \int_S \mathbf{n}(\mathbf{x}) \cdot [\hat{\mathbf{T}} \cdot \mathbf{u} - \mathbf{T} \cdot \hat{\mathbf{u}}] dS_x, \quad (7)$$

where  $\mathbf{n}$  is the unit outward normal to the fluid volume  $V$ ,  $S$  represents all surfaces bounding the domain (including a surface at infinity  $S_\infty$ ) and integration occurs with respect to  $\mathbf{x}$ .

Substituting the fundamental singular solutions (4) into the Reciprocal theorem, using (1), removing the arbitrary vector  $\mathbf{F}$ , and, for convenience, interchanging the labels  $\mathbf{x}$  and  $\mathbf{y}$  yields the integral equation

$$\frac{1}{\mu} \int_S \mathbf{n} \cdot \mathbf{T} \cdot \mathbf{J}(\mathbf{x}|\mathbf{y}) dS_y + \int_S \mathbf{n} \cdot \mathbf{K}(\mathbf{x}|\mathbf{y}) \cdot \mathbf{u} dS_y = \begin{cases} \mathbf{u}(\mathbf{x}) & \mathbf{x} \in V & (8a) \\ \frac{1}{2} \mathbf{u}(\mathbf{x}_s) & \mathbf{x}_s \in S & (8b) \\ 0 & \mathbf{x} \notin V & (8c) \end{cases}$$

The reader is warned that other sign/nomenclature conventions are in use and reminded that  $\mathbf{J}$  and  $\mathbf{K}$  are, respectively, symmetric and antisymmetric tensors with respect to  $\mathbf{r} = \mathbf{x} - \mathbf{y}$ . The factor of 1/2 arises from the jump in the value of the  $\mathbf{K}$  integral as the surface is crossed. This assumes that the surface is Lyapunov smooth, which requires that a local tangent to the interface exist everywhere; sharp corners, cusps or edges violate this assumption and must be treated separately [42].

Equation (8) relates the velocity field  $\mathbf{u}(\mathbf{x})$  at any point inside (or outside) a fluid volume, or on the boundary, to the velocity  $\mathbf{u}(\mathbf{x}_s)$  and traction  $\mathbf{n} \cdot \mathbf{T}(\mathbf{x}_s)$  on the bounding surface. For points on the boundary  $S$ , this is a Fredholm integral equation of the *second kind* for the velocity  $\mathbf{u}(\mathbf{x}_s)$  and of the *first kind* for the traction  $\mathbf{n} \cdot \mathbf{T}(\mathbf{x}_s)$ . Finally, equation (8c) is a useful identity for developing integral equations with a minimum number of unknowns (see §2.5-6).



There are alternative procedures for producing integral representations for Stokes flow problems. Rather than using the primary variables  $\mathbf{u}(\mathbf{x}_s), \mathbf{n} \cdot \mathbf{T}(\mathbf{x}_s)$  as in (8), an integral representation is developed in terms of non-physical single-layer or double-layer distributions. For example, the single-layer distribution is  $\mathbf{u}(\mathbf{x}) = \int_S \mathbf{J} \cdot \mathbf{q} \, dS_y$ , with a similar expression for the associated stress field. Unlike (8) which yields an integral equation of the first kind for the stress distribution  $\mathbf{n} \cdot \mathbf{T}$ , this alternative formulation yields an integral equation of the second kind. For problems involving rigid particles, the reader is referred to [22, 26]. The approach has also been applied to free-boundary problems [45, 55].

**2.4 Integral equation for a fluid-fluid interface**

Consider the case of a fluid drop, volume  $V_2$ , in an unbounded fluid, volume  $V_1$ , as shown in Fig. 2. The viscosity ratio is denoted by  $\lambda$ . If there is an imposed flow  $\mathbf{u}^\infty(\mathbf{x})$ , the basic equations are developed in terms of ‘disturbance variables’. Define for both fluids 1 and 2 the disturbance velocities  $\mathbf{u}'_{1,2}(\mathbf{x}) = \mathbf{u}_{1,2}(\mathbf{x}) - \mathbf{u}^\infty(\mathbf{x})$  and disturbance stresses  $\mathbf{T}'_{1,2}(\mathbf{x}) = \mathbf{T}_{1,2}(\mathbf{x}) - \mathbf{T}^\infty_{1,2}(\mathbf{x})$ .

Equation (8) holds for  $\mathbf{u}'$  in both fluids and only the fluid-fluid interface  $S_{int}$  need be considered. The enclosing boundary at large distances  $S_\infty$  may be neglected because of the  $1/r$  and  $1/r^2$  decay of the velocity and stress fields characteristic of disturbance Stokes flows [16]. Therefore, remembering that  $\mathbf{n}$  is directed from fluid 1 to fluid 2, we can write the two equations

$$-\frac{1}{\mu} \int_S \mathbf{n} \cdot \mathbf{T}'_1 \cdot \mathbf{J} \, dS_y - \int_S \mathbf{n} \cdot \mathbf{K} \cdot \mathbf{u}'_1 \, dS_y = \begin{cases} \mathbf{u}_1(\mathbf{x}) & \mathbf{x} \in V_1 \notin V_2 \\ \frac{1}{2} \mathbf{u}_1(\mathbf{x}_s) & \mathbf{x}_s \in S_{int} \\ 0 & \mathbf{x} \in V_2 \notin V_1 \end{cases} \quad (9)$$

and

$$\frac{1}{\lambda\mu} \int_S \mathbf{n} \cdot \mathbf{T}'_2 \cdot \mathbf{J} \, dS_y + \int_S \mathbf{n} \cdot \mathbf{K} \cdot \mathbf{u}'_2 \, dS_y = \begin{cases} 0 & \mathbf{x} \in V_1 \notin V_2 \\ \frac{1}{2} \mathbf{u}_2(\mathbf{x}_s) & \mathbf{x}_s \in S_{int} \\ \mathbf{u}_2(\mathbf{x}) & \mathbf{x} \in V_2 \notin V_1 \end{cases} \quad (10)$$

Multiplying equation (10) by  $\lambda$ , adding to (9), and using boundary conditions (2-3) yields

$$-\frac{1}{\mu} \int_{S_{int}} [\mathbf{n} \cdot \mathbf{T}'] \cdot \mathbf{J} \, dS_y - (1-\lambda) \int_{S_{int}} \mathbf{n} \cdot \mathbf{K} \cdot \mathbf{u}'_1 \, dS_y = \begin{cases} \mathbf{u}_1(\mathbf{x}) & \mathbf{x} \in V_1 \quad (11a) \\ \frac{1+\lambda}{2} \mathbf{u}_1(\mathbf{x}_s) & \mathbf{x}_s \in S_{int} \quad (11b) \\ \frac{\lambda}{\lambda\mu} \mathbf{u}_2(\mathbf{x}) & \mathbf{x} \in V_2 \quad (11c) \end{cases}$$

where the disturbance stress jump  $[\mathbf{n} \cdot \mathbf{T}'] = [\mathbf{n} \cdot \mathbf{T}] - [\mathbf{n} \cdot \mathbf{T}^\infty]$ . Alternatively, for problems with externally-imposed flows, it is more common to recast (11) in terms of the actual interfacial velocity and  $\mathbf{u}^\infty(\mathbf{x})$ . Using the identity

$$\int_{V_2} \nabla \cdot (\mathbf{T}^\infty \cdot \mathbf{J} + \mathbf{K} \cdot \mathbf{u}^\infty) \, dV = 0, \quad (12)$$

one obtains the widely applied integral equation

$$\mathbf{u}^\infty(\mathbf{x}_s) - \frac{1}{\mu} \int_{S_{int}} \llbracket \mathbf{n} \cdot \mathbf{T} \rrbracket \cdot \mathbf{J}(\mathbf{x}|\mathbf{y}) dS_y$$

$$- (1 - \lambda) \int_{S_{int}} \mathbf{n} \cdot \mathbf{K}(\mathbf{x}|\mathbf{y}) \cdot \mathbf{u}_1 dS_y = \begin{cases} \mathbf{u}_1(\mathbf{x}) & \mathbf{x} \in V_1 & (13a) \\ \frac{1+\lambda}{2} \mathbf{u}_1(\mathbf{x}_s) & \mathbf{x}_s \in S_{int}. & (13b) \\ \lambda \mathbf{u}_2(\mathbf{x}) & \mathbf{x} \in V_2 & (13c) \end{cases}$$

Rallison & Acrivos [63] first presented (13b) for the interfacial velocity and Pozrikidis [56] presented the complete form of (13) for the general case of points off the fluid-fluid interface. Power [49] showed that this integral equation has real eigenvalues corresponding to  $\lambda = 0$  or  $\infty$ , and that a unique and continuous solution exists for  $0 < \lambda < \infty$ .

For a given external flow and a known interface shape  $S_{int}$ , equation (11b) (or 13b) is a Fredholm integral equation of the second kind for the unknown disturbance velocity  $\mathbf{u}'(\mathbf{x}_s)$  (or the actual velocity  $\mathbf{u}(\mathbf{x}_s)$ ), which depends on the stress jump  $\llbracket \mathbf{n} \cdot \mathbf{T} \rrbracket$  across the deformable surface. The stress jump is a function of the interface shape, with dependence on the position, normal and curvature as given by equation (3). The above equations are the starting point for numerical investigation of numerous free-boundary problems. In addition, for a known interfacial velocity and interface shape, equations (11a, c) or (13a, c) determine the velocity at points off the surface and thus are useful for obtaining a detailed picture of the entire flow field.

The majority of studies assume that the interface shape is axisymmetric. In this case the azimuthal integration can be performed analytically, thereby reducing the surface integral to a line integral. The resulting kernels involve elliptic functions (*e.g.*, [31, 80]). For the case of equal fluid viscosities ( $\lambda = 1$ ), equations (11) and (13) simplify to an explicit expression for the interfacial velocity. If the problem sketched in Fig. 2 is bounded, for example, by a rigid planar boundary, a modified Green's function may be used to account for the rigid surface; only integration along the deformable interface  $S_{int}$  is required [3, 55].

In principle, either (11) or (13) and the kinematic condition are solved numerically in order to follow changes in the interface shape. However, before discussing details of the numerical solution, two additional applications are described for common free-boundary problems.

### 2.5 Application to a rigid particle near a fluid-fluid interface

Consider the slightly more complicated problem of a rigid buoyant sphere translating with unknown velocity  $\mathbf{U}_P$  normal to a deformable fluid-fluid interface in an otherwise quiescent fluid (Fig. 3a). Label the fluids 1 and 3 to be consistent with the related problems shown in Figs. 2 and 3b.

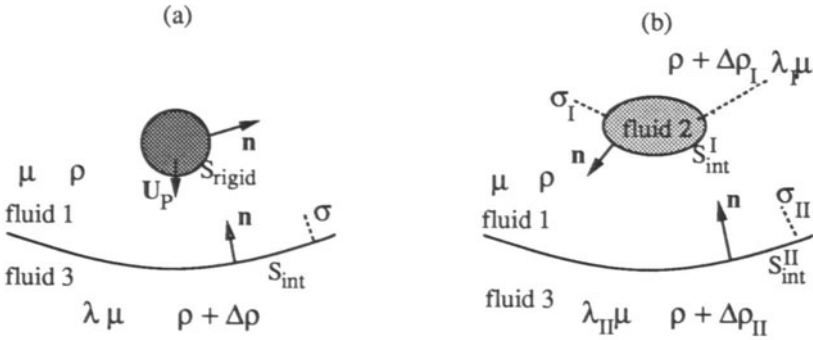


Figure 3: (a) Rigid sphere and (b) drop approaching an interface.

For fluid 1, use (8) and write four integral identities, two for field points on either of the bounding surfaces and two for points in either of the fluid volumes:

$$-\frac{1}{\mu} \int_{S_{int}} \mathbf{n} \cdot \mathbf{T}_1 \cdot \mathbf{J} \, dS_y - \int_{S_{int}} \mathbf{n} \cdot \mathbf{K} \cdot \mathbf{u}_1 \, dS_y \quad (14)$$

$$- \frac{1}{\mu} \int_{S_{rigid}} \mathbf{n} \cdot \mathbf{T}_1 \cdot \mathbf{J} \, dS_y = \begin{cases} \mathbf{u}_1(\mathbf{x}) & \mathbf{x} \in V_1 \\ \mathbf{U}_P & \mathbf{x}_s \in S_{rigid} \\ \frac{1}{2} \mathbf{u}_1(\mathbf{x}) & \mathbf{x}_s \in S_{int} \\ 0 & \mathbf{x} \in V_3 \end{cases}$$

The identity  $\int_{S_{rigid}} \mathbf{n} \cdot \mathbf{K} \, dS = -\frac{1}{2} \mathbf{I}$ , which follows from the Divergence Theorem and  $\nabla \cdot \mathbf{K} + \mathbf{I} \delta(\mathbf{r}) = 0$ , has been used to simplify the integral over  $S_{rigid}$ .

In a similar manner, for fluid 3 write the four equations

$$\frac{1}{\lambda\mu} \int_{S_{int}} \mathbf{n} \cdot \mathbf{T}_3 \cdot \mathbf{J} \, dS_y + \int_{S_{int}} \mathbf{n} \cdot \mathbf{K} \cdot \mathbf{u}_3 \, dS_y = \begin{cases} 0 & \mathbf{x} \in V_1 \\ 0 & \mathbf{x}_s \in S_{rigid} \\ \frac{1}{2} \mathbf{u}_3(\mathbf{x}) & \mathbf{x}_s \in S_{int} \\ \mathbf{u}_3(\mathbf{x}) & \mathbf{x} \in V_3 \end{cases} \quad (15)$$

Multiplying (14) by  $\lambda$ , adding to (15), and again using the boundary conditions (2-3) of continuous velocity and the stress jump  $[[\mathbf{n} \cdot \mathbf{T}]]$  yields

$$-\frac{1}{\mu} \int_{S_{int}} [[\mathbf{n} \cdot \mathbf{T}]] \cdot \mathbf{J} \, dS_y - (1 - \lambda) \int_{S_{int}} \mathbf{n} \cdot \mathbf{K} \cdot \mathbf{u}_1 \, dS_y \quad (16)$$

$$- \frac{1}{\mu} \int_{S_{rigid}} \mathbf{n} \cdot \mathbf{T}_1 \cdot \mathbf{J} \, dS_y = \begin{cases} \mathbf{u}_1(\mathbf{x}) & \mathbf{x} \in V_1 \\ \mathbf{U}_P(\mathbf{x}) & \mathbf{x}_s \in S_{rigid} \\ \frac{1 + \lambda}{2} \mathbf{u}_1(\mathbf{x}) & \mathbf{x}_s \in S_{int} \\ \lambda \mathbf{u}_3(\mathbf{x}) & \mathbf{x} \in V_3 \end{cases}$$

The problem statement is completed by imposing the integral force constraint on the buoyantly rising particle,  $\int_{S_{rigid}} \mathbf{n} \cdot \mathbf{T}_1 dS = \text{constant}$ , where the spherical particle is assumed not to rotate.

Equation (16) and the kinematic condition represent coupled equations for the unknowns  $\mathbf{u}_1(\mathbf{x}_s \in S_{int})$ ,  $\mathbf{n} \cdot \mathbf{T}_1(\mathbf{x}_s \in S_{rigid})$ , the rise speed of the particle  $\mathbf{U}_P$ , and the interface shape. If the particle translation velocity is specified, the force constraint is not applied. Also, following determination of the unknowns, the velocity field off the interface can be calculated. This formulation involves fewer unknowns than previous studies, which required solving for the actual stress distribution along the interface *e.g.*, [31].

The problem formulation as an integral equation may be generalized easily to allow for an arbitrarily-shaped rigid particle by introducing an unknown particle rotation rate and imposing a torque-free constraint.

## 2.6 Generalization to multiple fluid-fluid interfaces

We now extend the discussion given in §2.4-5 to examine free-boundary problems involving multiple fluid-fluid interfaces. Consider the motion of a deformable drop moving normal to a deformable interface, sketched in Fig. 3b. Numerical results for this problem are presented in §4.

In many previous applications of boundary integral methods to problems with multiple deformable interfaces, both the interfacial velocity and interfacial stresses remained as unknowns in the final equations, *e.g.*, [71]. However, by making full use of the integral relation defined by equation (8), integrals involving the *actual stress* can be replaced by integrals involving only velocities and *stress jumps* at each of the deformable interfaces. This formulation avoids solving an integral equation of the first kind for the stress.

The fluid velocities  $\mathbf{u}_1, \mathbf{u}_2$  and  $\mathbf{u}_3$  (see Fig. 3b) can be written as integrals over all bounding surfaces using equation (8). Using a superscript *I* or *II* to distinguish the interfaces, the derivation presented in §2.5 can be generalized to show that the velocity field at any point along either of the two interfaces,  $S_{int}^I$  and  $S_{int}^{II}$ , or in either of the three fluid volumes is given concisely by

$$\begin{aligned}
 & -\frac{1}{\mu} \int_{S_{int}^I} [\mathbf{n} \cdot \mathbf{T}^I] \cdot \mathbf{J} dS_y - (1 - \lambda_I) \int_{S_{int}^I} \mathbf{n} \cdot \mathbf{K} \cdot \mathbf{u}_1^I dS_y - \frac{1}{\mu} \int_{S_{int}^{II}} [\mathbf{n} \cdot \mathbf{T}^{II}] \cdot \mathbf{J} dS_y \\
 & - (1 - \lambda_{II}) \int_{S_{int}^{II}} \mathbf{n} \cdot \mathbf{K} \cdot \mathbf{u}_1^{II} dS_y = \begin{cases} \mathbf{u}_1(\mathbf{x}) & \mathbf{x} \in V_1 & (17a) \\ \lambda_I \mathbf{u}_2(\mathbf{x}) & \mathbf{x} \in V_2 & (17b) \\ \lambda_{II} \mathbf{u}_3(\mathbf{x}) & \mathbf{x} \in V_3 & (17c) \\ \frac{1 + \lambda_I}{2} \mathbf{u}_1^I(\mathbf{x}_s) & \mathbf{x}_s \in S_{int}^I & (17d) \\ \frac{1 + \lambda_{II}}{2} \mathbf{u}_1^{II}(\mathbf{x}_s) & \mathbf{x}_s \in S_{int}^{II} & (17e) \end{cases}
 \end{aligned}$$

where  $[\mathbf{n} \cdot \mathbf{T}^I]$  and  $[\mathbf{n} \cdot \mathbf{T}^{II}]$  denote the stress jumps across the two fluid-fluid interfaces. Equations (17d) and (17e) represent two coupled integral equations for the unknown interfacial velocities  $\mathbf{u}_1^I$  and  $\mathbf{u}_1^{II}$  and involve only the two stress jumps.

### 3. NUMERICAL IMPLEMENTATION

This section summarizes standard numerical solution procedures for treating the free-boundary problem. The discussion refers to the simplest case of deformation of a single interface (§2.4), but can be easily extended to multiple interfaces.

In many problems, the interface is continually deforming and the rate and degree of deformation (*e.g.*, drops breaking into smaller drops) are common topics of study. Numerical approaches for these transient problems will be addressed here; other approaches for determining steady shapes are discussed in [53, 81].

The integral equations derived in §2 are linear equations for the interfacial velocity  $\mathbf{u}(\mathbf{x}_s)$ , but are non-linear when the unknown shape is included. The interface evolves according to  $d\mathbf{x}_s/dt = \mathbf{u}(\mathbf{x}_s)$ . Representing the interface at  $N$  discrete points  $\mathbf{x}_s^i$ , the numerical problem is to determine the evolution of  $\mathbf{x}_s^i$  according to the system of equations

$$\frac{d\mathbf{x}_s^i(t)}{dt} = \mathbf{u}(\mathbf{x}_s^i) = \mathcal{F} \left\{ \mathbf{x}_s^i, \frac{d\mathbf{x}_s^i(t)}{dt} \right\}, \quad i = 1, \dots, N \quad (18)$$

where the non-linear functional  $\mathcal{F}$  (*e.g.*, equation 11b) depends on the unknown interface shape (hence involves knowledge of the surface normal and curvature) and the unknown interfacial velocity.

Almost all solutions of this problem *linearize* (18) by relaxing the kinematic condition so that the velocity calculation *decouples* from the determination of the unknown shape. There are three primary steps in this *iterated* solution procedure: (i) describe the deformed interface; (ii) calculate the surface velocity for a given shape by solving the second kind integral equation (11b) or (13b); (iii) march the interface shape forward in time using the kinematic condition. These three items are now discussed.

#### 3.1 Interface description

Free boundary problems, which involve interfacial tension, must accurately determine the position, normal and *curvature* of the interface. Thus the deforming interface shape ideally should have a twice continuously differentiable representation.

Axisymmetric or two-dimensional interface shapes have been studied extensively. In such cases it is common to represent the surface at  $N$  discrete nodal points. The position, normal, and curvature may be evaluated either directly at the nodes using a finite difference scheme [63], or at

inter-nodal locations using interpolation. Examples of the latter use either a local interpolating function over a boundary element (piecewise polynomial interpolation [42], local circular arc approximation [45]), or a global interpolation involving all nodes (cubic spline *e.g.* [15]). Potential difficulties with highly distorted shapes, such as multi-valued representations and/or infinite derivatives, may be eliminated using an arclength parameterization scheme. The interface is represented using two cubic splines with interface arclength as the independent variable [68] (*e.g.*,  $r = r(s), z = z(s)$ ) where  $r, z$  are a cylindrical coordinate representation of the surface and  $s$  is the arclength). Investigations of three-dimensional surface deformation are still in their early stages and only modest distortions have been computed, primarily in the context of drop deformation in simple shear flow [13, 25, 60, 61].

### 3.2 Calculation of the interfacial velocity (solution of the integral equation)

Given a description of the interface, the integral equation is reduced to a system of linear algebraic equations for the interfacial velocities  $\mathbf{u}(\mathbf{x}_s^i)$ , following one of several possible numerical techniques [1, 21, 26]. The most commonly used is a collocation technique, whereby the unknown velocity is discretized and approximated using a local polynomial interpolation, a quadrature scheme is introduced and the integral equation is enforced at the discrete points.

The resulting system of equations produces a dense matrix which may be solved by a direct method such as Gaussian elimination. Pozrikidis [60] has shown that iterative methods may be more efficient because a good initial guess for the value of the unknown is available from the previous time step. The potential time-savings may be important as larger and more complex problems are studied.

### 3.3 Interface evolution

The interface shape is updated using the kinematic condition. For example, the marker points may be marched forward using the actual surface velocity (a Lagrangian representation). This procedure tends to sweep points tangent to the surface even if only small shape changes actually occur. Consequently, frequent redistribution of points is necessary. For this case it is straightforward to implement a multi-step integrator.

Alternatively, an Eulerian view point can be taken [3]. In practice, points on the interface are moved in a direction normal to the interface using the normal projection of the surface velocity,  $d\mathbf{x}_s/dt = (\mathbf{n} \cdot \mathbf{u}(\mathbf{x}_s)) \mathbf{n}$ . This has the advantage that marker points tend to remain evenly distributed. A simple explicit Euler method is generally used because it is unclear how best to implement higher order integrators.

### 3.4 Miscellaneous remarks

The kernels  $\mathbf{J}$  and  $\mathbf{K}$  have integrable singularities (at  $\mathbf{y} = \mathbf{x}_s$ ) and some care is needed in the numerical integration. Common procedures either cut out a region surrounding the singularity and perform the local integration analytically [63] or subtract the singularity directly in the numerical approximation of the integral [33].

If the interface extends to infinity (*e.g.*, Fig. 3), a simple approximation to solve the integral equation is to truncate the interface at a finite distance and verify that the truncation distance is adequate [31]. Alternatively, the interface may be extrapolated with an approximate curve, and the integration to infinity treated using an appropriate quadrature scheme [36].

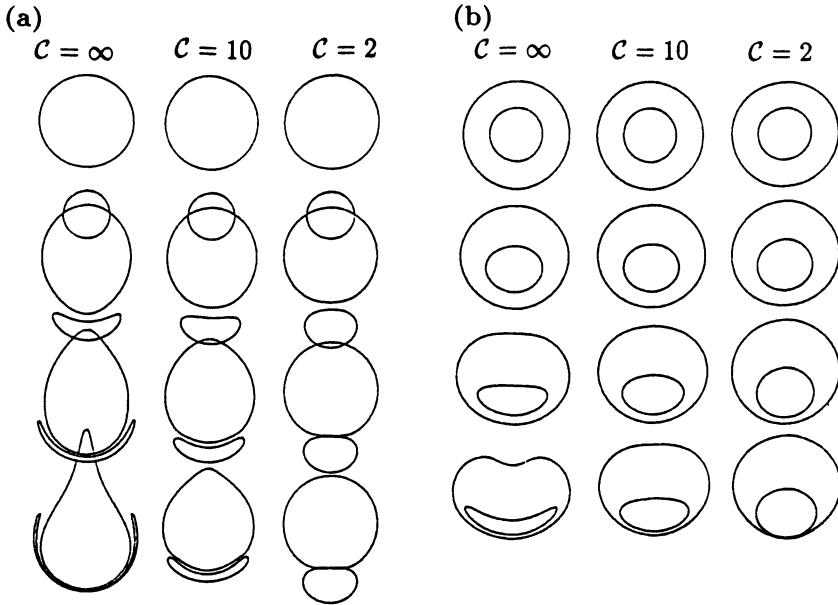
In most studies of drops, the constant volume of the drop is not imposed, but rather volume changes are used as a measure of the accuracy of the numerical method. For  $\lambda = 1$ , the volume changes typically are insignificant. However, most researchers report noticeable volume changes for low viscosity ratios,  $\lambda \lesssim 0.1$ , and may resort to a shape rescaling in order to continue the simulations for long times.

There have been one or two preliminary attempts, *e.g.* [79], to solve (18) using a fully implicit (Newton) scheme. In addition, parallel processing has been recently applied to boundary element calculations of suspensions of rigid spheres in Stokes flows [23]. Future work in both of these directions would be useful.

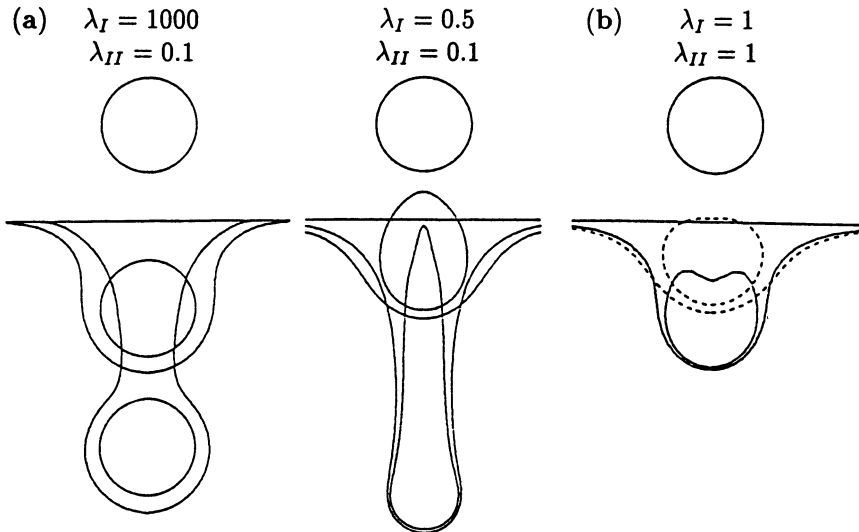
## 4. APPLICATIONS TO SELECTED PROBLEMS

In this section we show examples of the deformation of buoyant drops in axisymmetric three-fluid systems to illustrate the use of the boundary integral approach: (i) two drops translating parallel to their line-of-centers, (ii) translating double emulsion drops, and (iii) a drop approaching an interface. For a given initial configuration and a constant interfacial tension five dimensionless numbers characterize the system: two viscosity ratios,  $\lambda_I$  and  $\lambda_{II}$  (see Fig. 3b), two Capillary numbers,  $C_I = \mu u / \sigma_I$  and  $C_{II} = \mu u / \sigma_{II}$ , where  $u$  is a representative settling speed in an unbounded fluid, and a buoyancy parameter  $B = \Delta \rho_{II} / \Delta \rho_I$ .

The integral equations are solved using a collocation technique. The interfaces are described by twice continuously differentiable taut cubic splines parameterized in terms of arclength [12]. To minimize numerical error collocation points are concentrated in regions where the interface separations are smallest and are redistributed at each time step. Typically 30-100 collocation points and 400-1000 Euler time steps are used. Computation times ranged from 20 minutes to 12 hours on a Sparc 2. The simulations are stopped when the interface separation becomes smaller than the



**Figure 4:** Time-dependent deformation of (a) two buoyant drops and (b) double emulsion drops for different values of interfacial tension,  $\sigma$ . The Capillary number  $C = \mu u / \sigma$ . The fluids have the same viscosity ( $\lambda_I = \lambda_{II} = 1$ ) and both interfacial tensions are equal ( $\sigma_I = \sigma_{II}$ ). The buoyancy parameter  $B = 1$ .



**Figure 5:** Time-dependent deformation of a buoyant drop passing through an interface showing the effect of (a) drop viscosity ( $\lambda_I \mu$ ) and (b) lower fluid viscosity ( $\lambda_{II} \mu$ ). The buoyancy parameter  $B = 0.2$ , and there is no interfacial tension ( $C = \infty$ ).



local node spacing.

Fig. 4 illustrates the time-dependent deformation of two buoyant drops or double emulsion drops translating parallel to their line-of-centers for different values of the interfacial tension. Each column shows a sequence of drop shapes at different times. For low values of the interfacial tension, the drops may become highly deformed. Fig. 5 shows the effect of fluid viscosity on drops approaching a fluid-fluid interface. Drop viscosity plays an important role in the rate and nature of film drainage while the viscosity of the lower fluid influences the mode of deformation: either a tail or a cavity develops at the back of the drop. Figs. 4 and 5 display the ability of the numerical method to track successfully large deformations.

## 5. SUMMARY

Integral equations of the second kind for the velocity along a fluid-fluid interface in low Reynolds number flows have been discussed. The development is extended in a straightforward manner to problems with multiple interfaces. In all cases, only the interfacial velocities and stress jumps across fluid-fluid interfaces, and the stress distributions and velocities along rigid surfaces, appear as unknowns. A summary of the numerical solution of these free-boundary problems has been given and an example shown of the deformation of buoyant drops in systems with other drops or interfaces.

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We have made every effort to provide a thorough overview of low Reynolds number free-boundary problems solved using the boundary integral method, but apologize in advance for any references that we may have missed. Also, we encourage any additions to the lists given in Fig.1 or Table 1 to be brought to our attention (E-mail: has@stokes.harvard.edu).

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