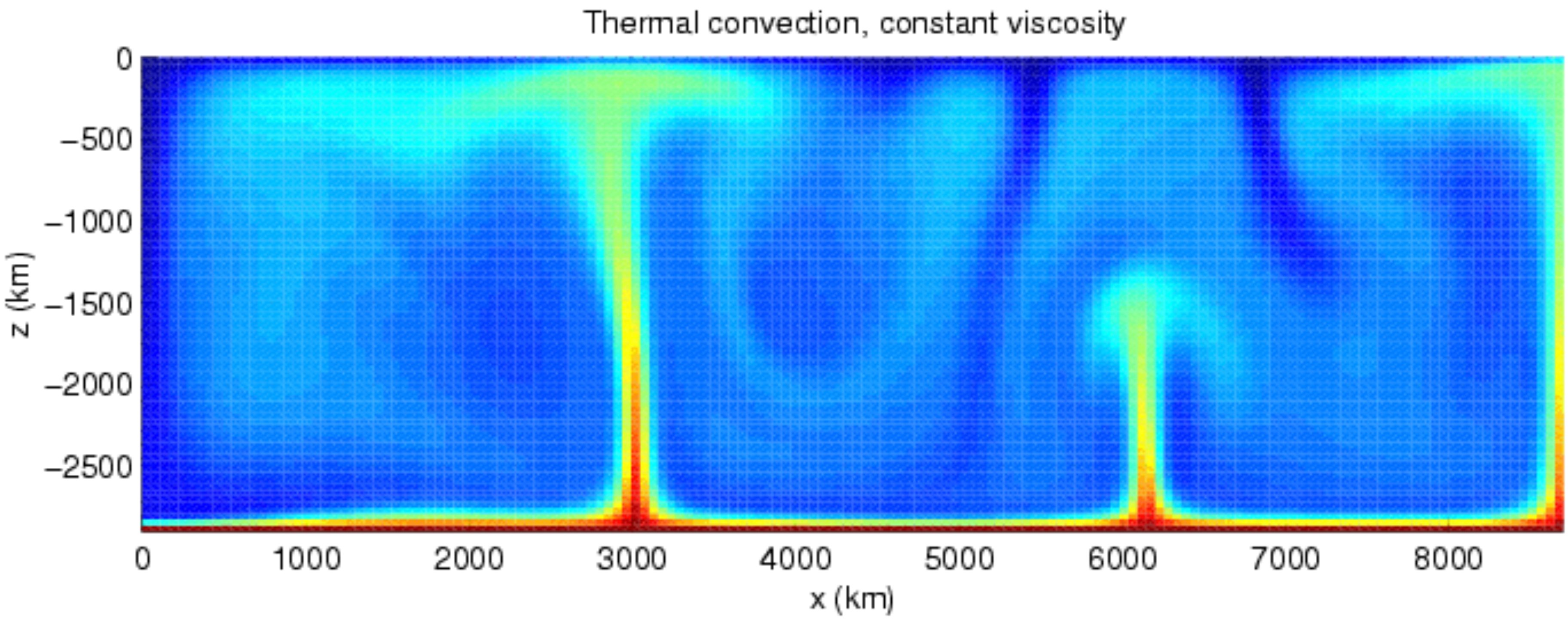
Geodynamics I
Basics of Thermal Convection

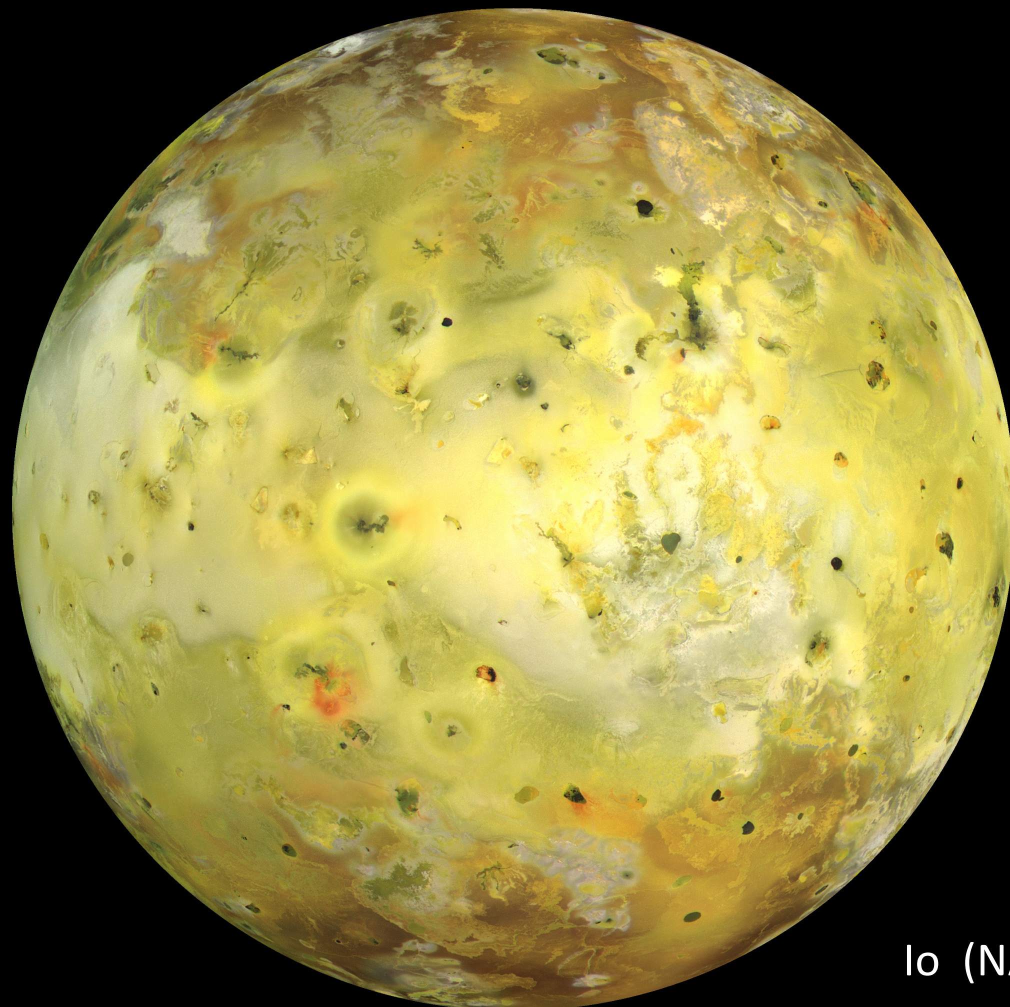


Quelle & Schmelling (2002)

Bruce Buffett, UC Berkeley

Presented by Michael Manga, UC Berkeley

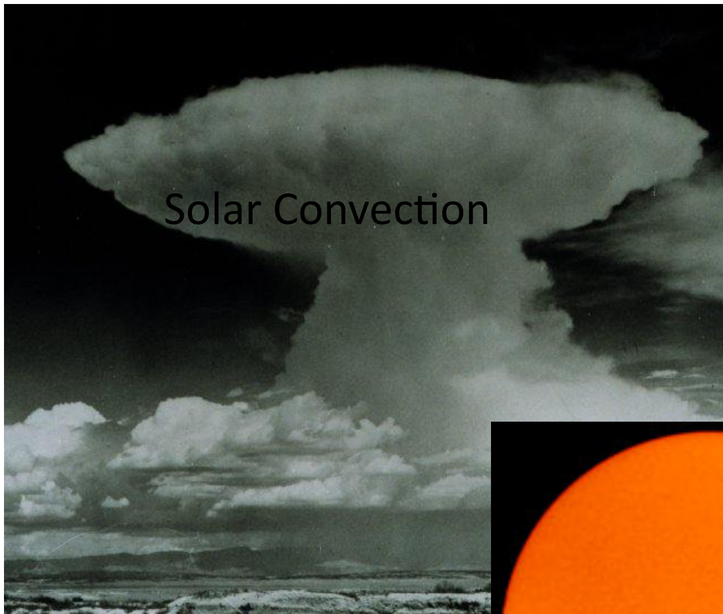
Other Bodies



Io (NASA)

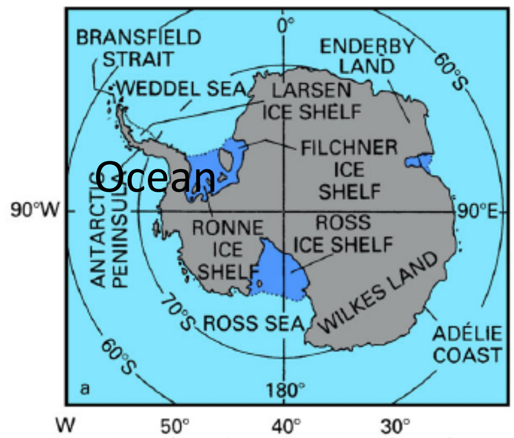
Other Contexts

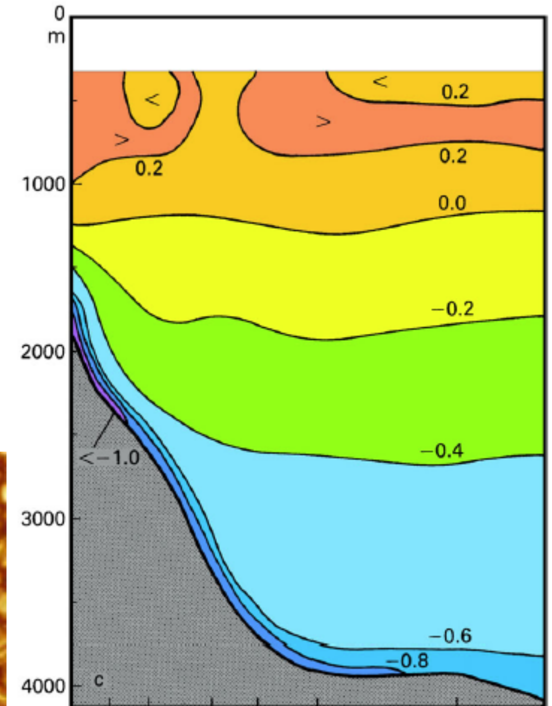
Atmosphere



NOAA

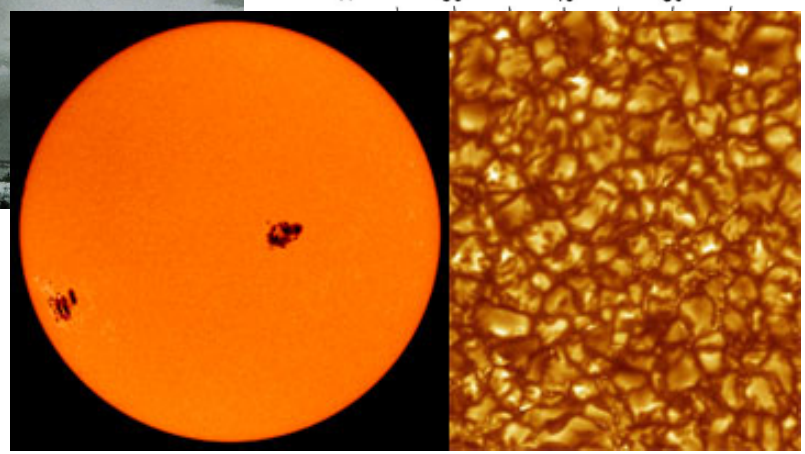
Ocean





Rahmstorf 2006

Solar Convection



Abbett et al. 2004

Wide range of outcomes are predicted by basic equations

Outline

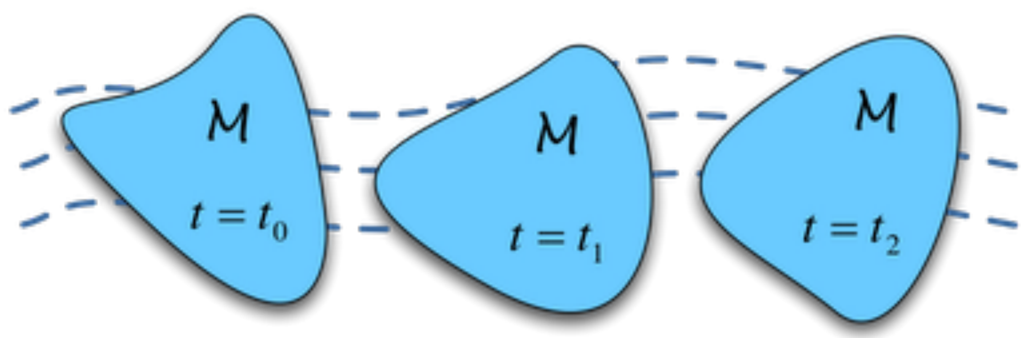
1. A brief overview of the governing equations
2. Introduction to dimensionless numbers
3. Onset of convection (elements of stability)
4. The boundary layer model
5. Scaling relations and thermal histories, and some complications

Please ask questions as issues become unclear or you have other questions

Conservation of mass

Some Tools

Fluid Parcel (mass M or volume V)

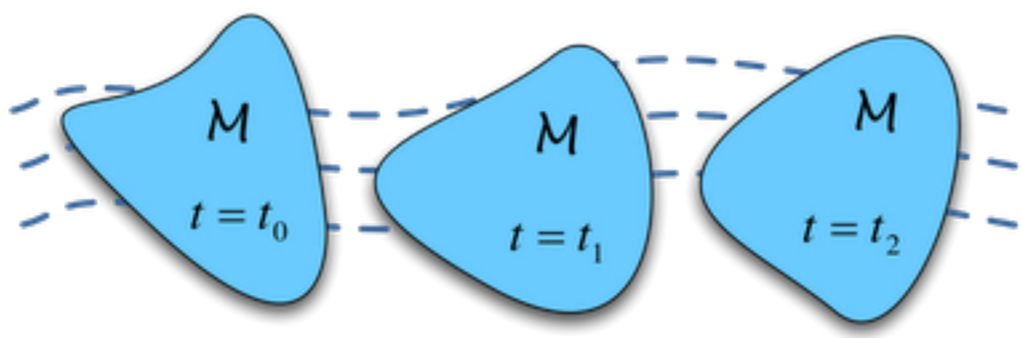


1. Time Derivative (e.g. acceleration **a**)

$$\mathbf{a} = \frac{d\,\mathbf{v}(\mathbf{x},t)}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial t}$$

Some Tools

Fluid Parcel (mass M or volume V)

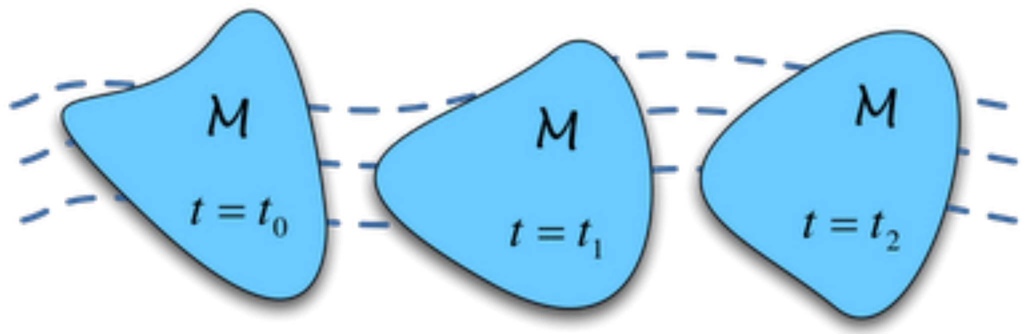


1. Time Derivative (e.g acceleration **a**)

$$\mathbf{a} = \frac{d\,\mathbf{v}(\mathbf{x}, t)}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}$$

Some Tools

Fluid Parcel (mass M or volume V)



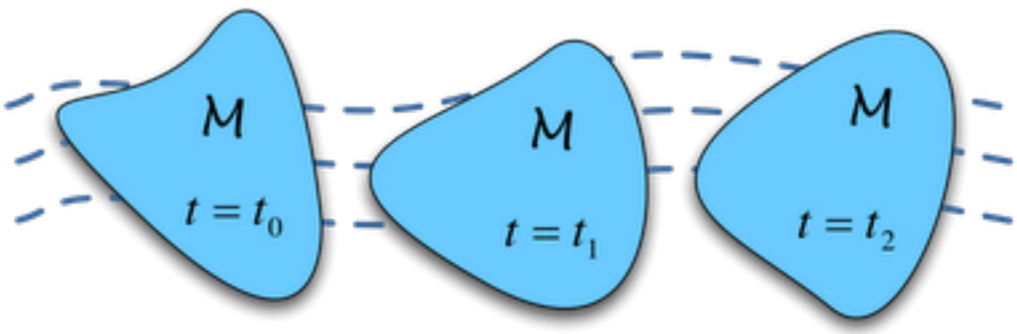
1. Time Derivative (e.g acceleration **a**)

$$\mathbf{a} = \frac{d\,\mathbf{v}(\mathbf{x}, t)}{dt} = \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)\mathbf{v} = \frac{D}{Dt}\mathbf{v}$$

material derivative

Another Tool

Fluid Parcel (mass M or volume V)

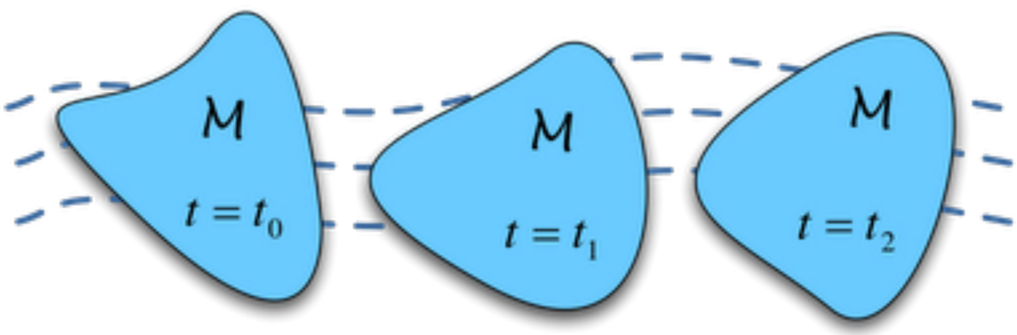


2. Reynolds Transport Theorem

$$\frac{d}{dt}\int_{V(t)} f\,dV = \int_{V(t)} \left[\frac{Df}{Dt} + f(\nabla\cdot\mathbf{v})\right] dV = 0$$

An Example

Mass of parcel $$M = \int_{V(t)} \rho \, dV$$



$$\frac{d}{dt} \int_{V(t)} \rho \, dV = \int_{V(t)} \left[\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) \right] dV = 0$$

Conservation of Mass

Conservation of mass requires

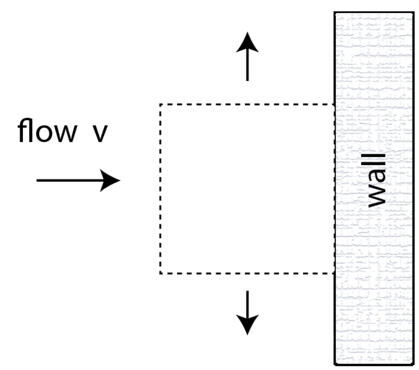
$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0$$

Equivalent form (substitute for Dρ/Dt)

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho(\nabla \cdot \mathbf{v}) = 0$$

Incompressible flow

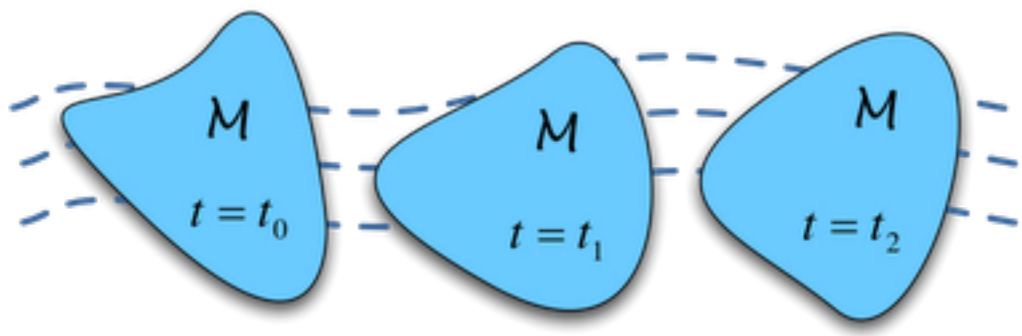
$$\nabla \cdot \mathbf{v} = 0$$



Conservation Equations

Conservation of Momentum

$$\mathbf{p} = \int_{V(t)} \rho \mathbf{v} \, dV$$



Newton's 2nd Law

$$\frac{d}{dt} \int_{V(t)} \rho \mathbf{v} \, dV = \mathbf{F}$$

total force on parcel V(t)

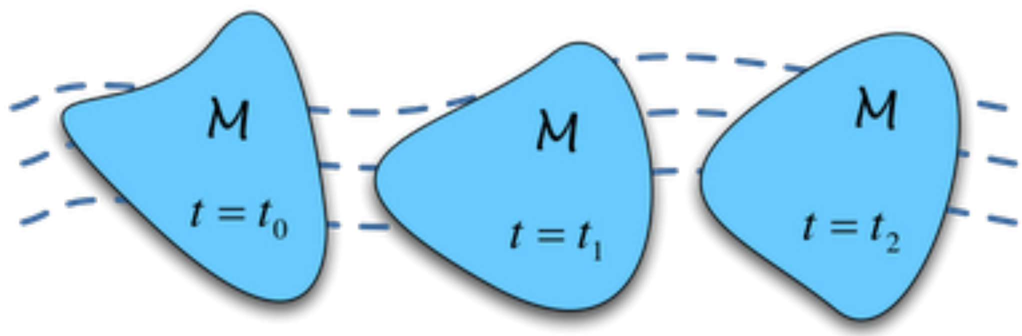
(e.g. gravity, pressure, viscous drag, etc)

Conservation of momentum

Conservation Equations

Conservation of Heat*

$$H = \int_{V(t)} \rho C_p T \, dV$$



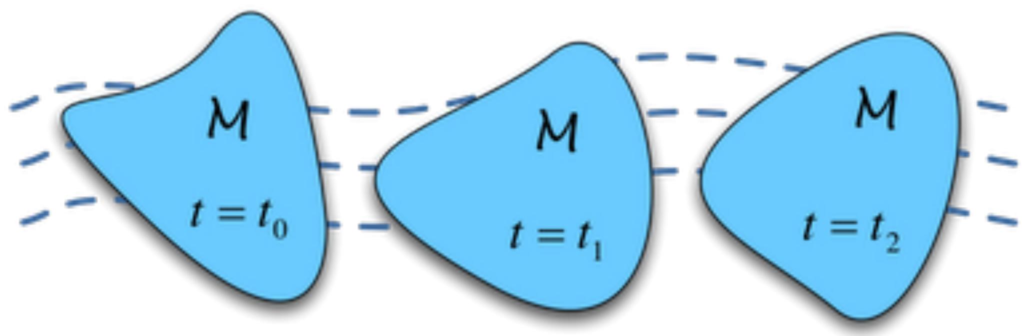
$$\frac{d}{dt} \int_{V(t)} \rho C_p T \, dV = - \int_{S(t)} \mathbf{q} \cdot \mathbf{dS} + \int_{V(t)} R \, dV$$

$$(\mathbf{q} = -k \nabla T)$$

conduction across surface S(t)

Conservation Equations

Conservation of Heat* $H = \int_{V(t)} \rho C_p T \, dV$



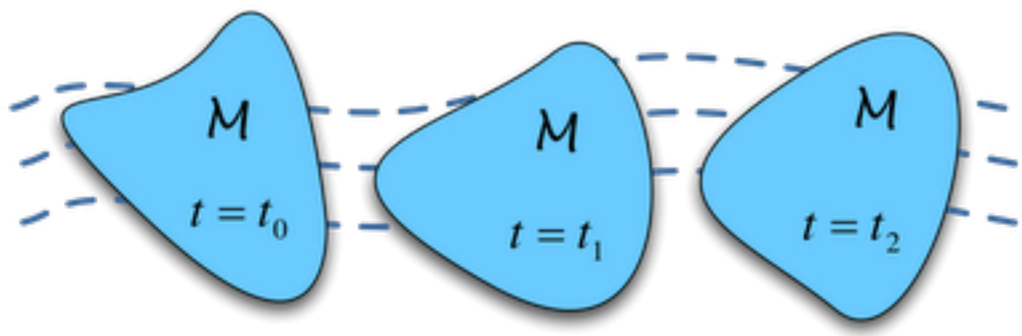
$$\frac{d}{dt} \int_{V(t)} \rho C_p T \, dV = - \int_{V(t)} \nabla \cdot \mathbf{q} \, dV + \int_{V(t)} R \, dV$$

$$(\mathbf{q} = -k \nabla T)$$

Conservation Equations

Conservation of Heat*

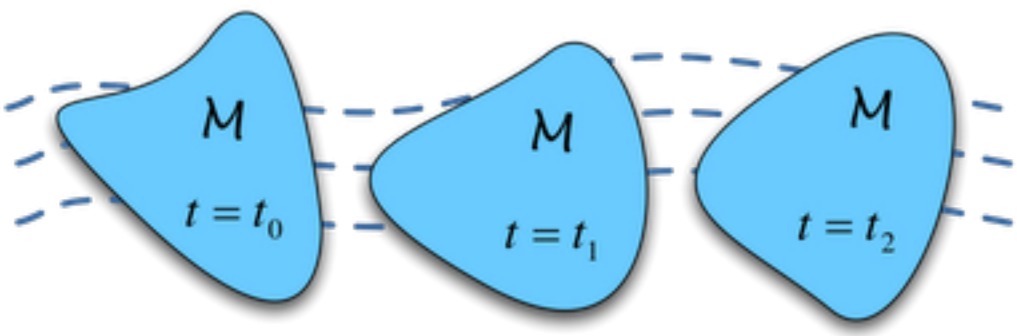
$$H = \int_{V(t)} \rho C_p T \, dV$$



$$\int_{V(t)} \left(\rho C_p \frac{DT}{Dt} + \rho C_p T (\nabla \cdot \mathbf{v}) \right) dV = \int_{V(t)} (\nabla \cdot k \nabla T + R) \, dV$$

Conservation Equations

Conservation of Heat*
$$H = \int_{V(t)} \rho C_p T \, dV$$



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$$(\mathbf{q} = -k \nabla T)$$

* assumes constant ρ and C_p

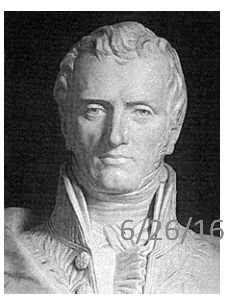
Summary for Incompressible Fluid

mass $$\frac{D\rho}{Dt} = 0 \quad \longrightarrow \quad \nabla \cdot \mathbf{v} = 0$$

momentum $$\rho \frac{D\mathbf{v}}{Dt} = -\nabla P + \rho \mathbf{g} + \eta \nabla^2 \mathbf{v}$$

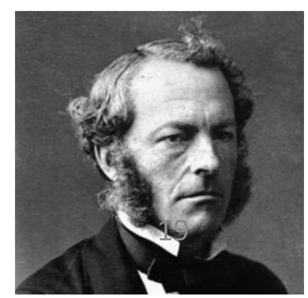
heat $$\rho C_p \frac{DT}{Dt} = k \nabla^2 T + R$$

Momentum equation is often called Navier-Stokes equation



Claude
Navier

George
Stokes



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heat $$\rho C_p \frac{DT}{Dt} = k\nabla^2 T + R$$

Joseph Boussinesq

Equation of state: relate density variations to changes in temperature

$$\frac{1}{\rho}\frac{\partial \rho}{\partial T} = -\alpha$$

thermal expansion

Summary for Incompressible Fluid

mass $$\frac{D\rho}{Dt} = 0 \longrightarrow \nabla \cdot \mathbf{v} = 0$$

momentum $$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla P + \rho \mathbf{g} + \eta \nabla^2 \mathbf{v}$$

heat $$\rho C_p \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = k \nabla^2 T + R$$

+ Boundary conditions:
  - free slip, v=0
  - heat flow, fixed T

Digression on "Viscous" Force

Force $\mathbf{f} = \nabla \cdot \tau$ where τ is deviatoric stress tensor

Newtonian fluid $\tau = 2\eta\dot{\epsilon}$ where strain rate $\dot{\epsilon} = \frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)$

$\Rightarrow$ $\mathbf{f} = \eta \nabla^2 \mathbf{v}$

More generally

$$\dot{\epsilon} = f(T, P, \tau, ...)\tau$$

Other variables: grain size, volatiles, partial melt, deformation history

Mineral Physics lectures 2, 4 and tutorial 2

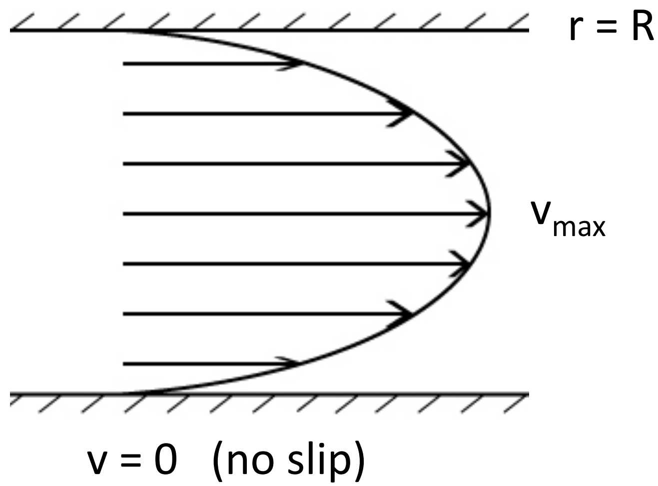
2. Dimensionless Numbers

Define dimensionless variables

$x' = x / R$

$v' = v / v_{max}$

$t' = t / \tau$



Navier-Stokes (in SI units)

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla P + \eta\nabla^2\mathbf{v}$$

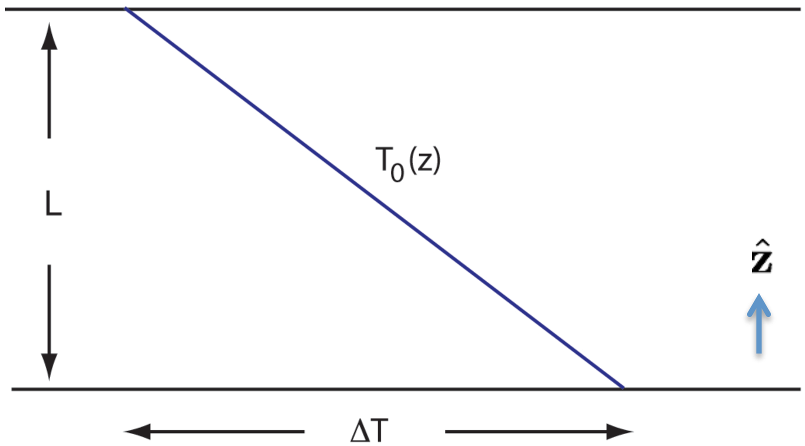
Change variables to x', t', v'

$$\left(\frac{\partial \mathbf{v}'}{\partial t'} + \mathbf{v}'\cdot\nabla\mathbf{v}'\right) = -\nabla P' + \frac{1}{Re}\nabla^2\mathbf{v}'$$

$$Re = \frac{v_{max}R}{(\eta/\rho)} = \frac{v_{max}R}{\nu}$$

Reynolds number

3. Convection Problem



Dimensionless variables

x' = x / L

T' = T / ΔT

t' = t / τ where $\tau = L^2/\kappa$

v' = v / (L/τ)

Change variables to dimensionless quantities

1. $\nabla \cdot \mathbf{v}' = 0$

2. $\frac{1}{Pr}\left(\frac{\partial \mathbf{v}'}{\partial t'} + \mathbf{v}' \cdot \nabla \mathbf{v}'\right) = -\nabla P' + RaT'\hat{\mathbf{z}} + \nabla^2 \mathbf{v}'$

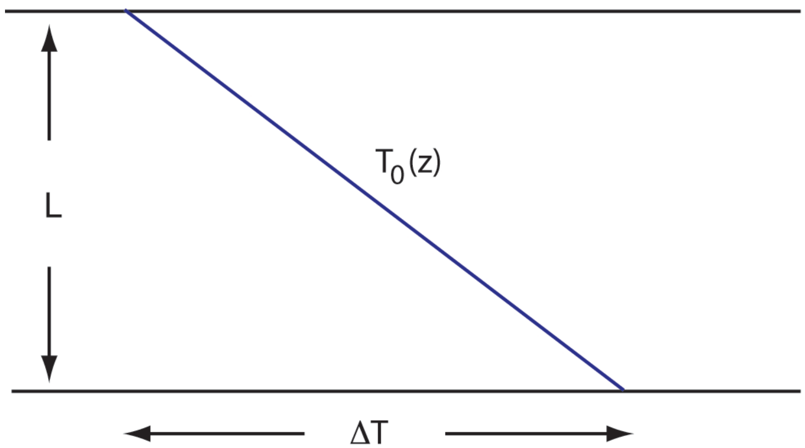
3. $\frac{\partial T'}{\partial t'} + \mathbf{v}' \cdot \nabla T' = \nabla^2 T' + R'$

Dimensionless numbers

$$Pr = \frac{\nu}{\kappa}$$

$$Ra = \frac{\alpha g \Delta T L^3}{\kappa \nu}$$

Convection Problem



Dimensionless variables

$x' = x / L$

$T' = T / \Delta T$

$t' = t / \tau$ where $\tau = L^2/\kappa$

$v' = v / (L/\tau)$

Change variables to dimensionless quantities

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3. $\frac{\partial T'}{\partial t'} + \mathbf{v}' \cdot \nabla T' = \nabla^2 T' + R'$

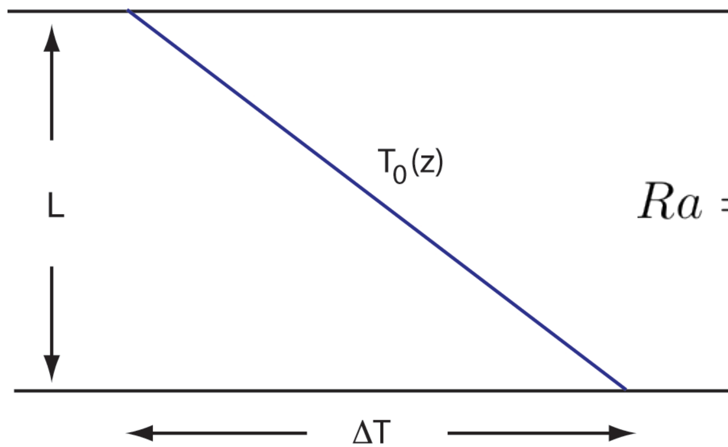
Dimensionless numbers

$Pr = \frac{\nu}{\kappa}$ ~ 10^{24} (mantle)

$Ra = \frac{\alpha g \Delta T L^3}{\kappa \nu}$

Onset of Convection

When does convection begin?



$$Ra = \frac{\alpha g \Delta T L^3}{\kappa \nu}$$

Consider time evolution of a small perturbation in an initially conductive state

$$T(x,y,z,t) = T_0(z) + \delta T(x,y,z)\, e^{\sigma t}$$

$$\mathbf{v}(x,y,z,t) = \delta \mathbf{v}(x,y,z) e^{\sigma t}$$

Substitute into (linearized) equations and solve for growth rate σ

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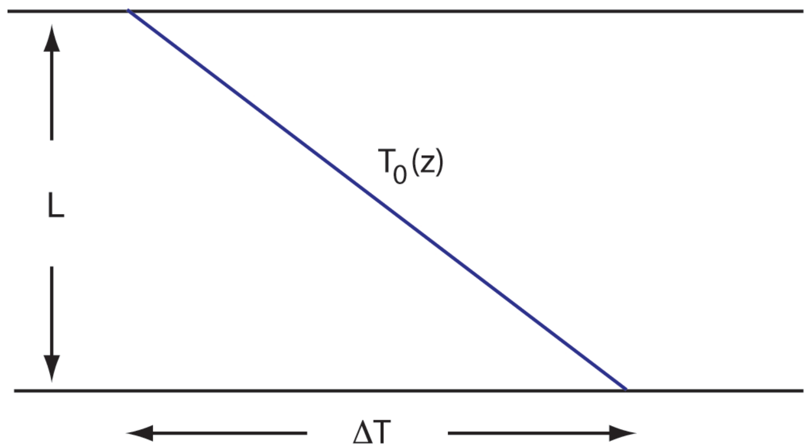

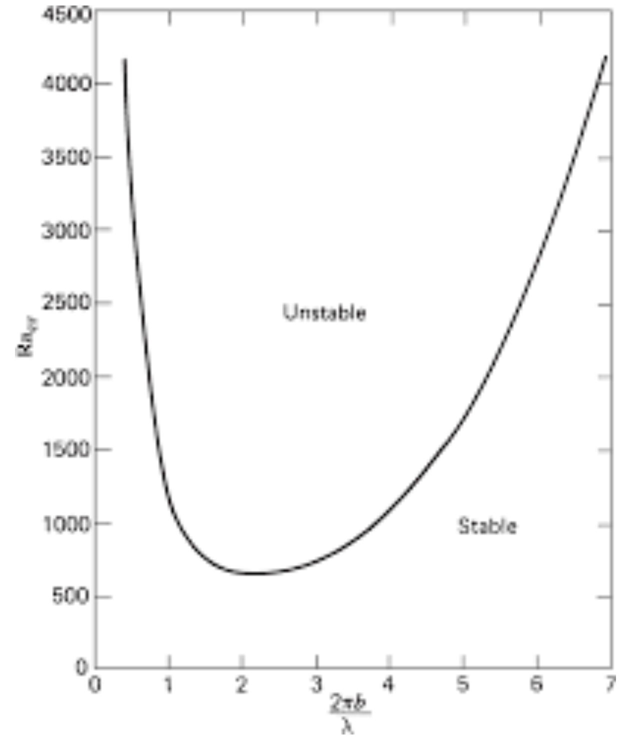

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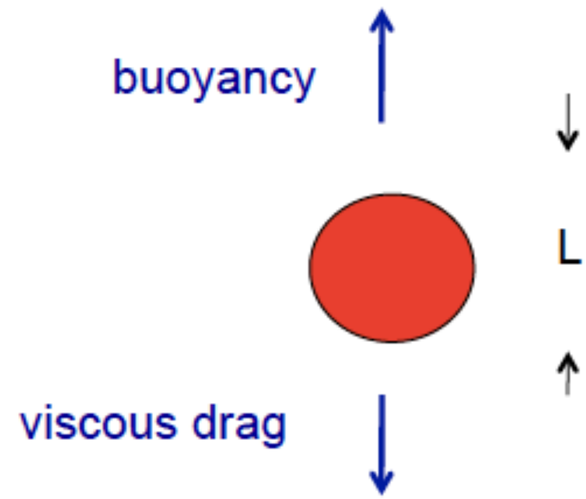
Rayleigh Number Ra



Velocity of Parcel $v \approx \Delta\rho g L^2/\eta$

For hot fluid $|\Delta\rho| = \rho\alpha\Delta T$

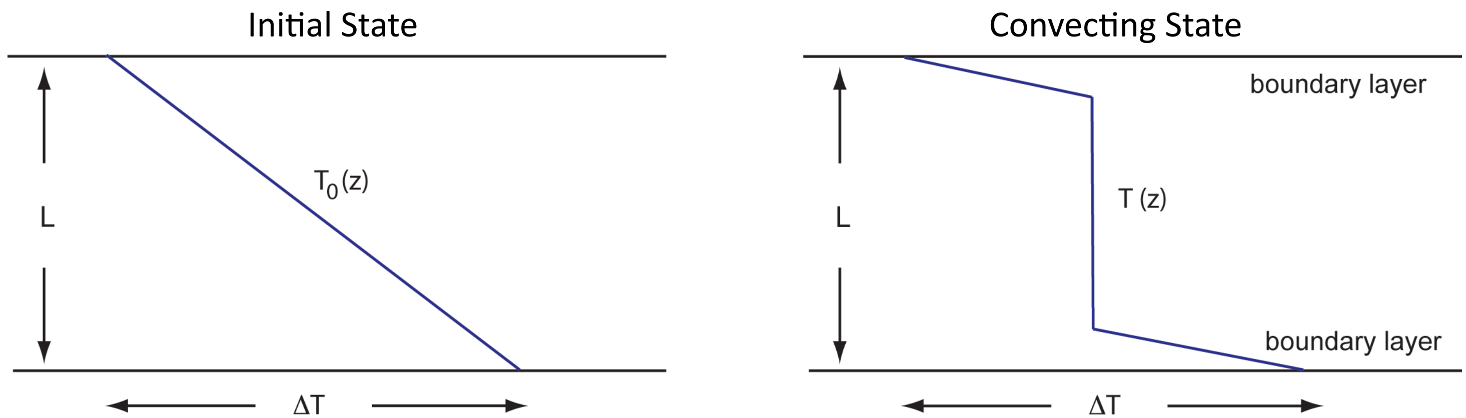
Ratio of conduction to advection time?

$$\frac{\tau_c}{\tau_a} = \frac{vL}{\kappa} = \frac{\rho\alpha g\Delta T L^3}{\kappa\eta}$$

(Rayleigh number)

e.g. L = 2900 km, $\Delta T \sim 3000$ K, Ra ~ 10^8 (critical $Ra_c \sim 10^3$)

4. Boundary Layer Theory



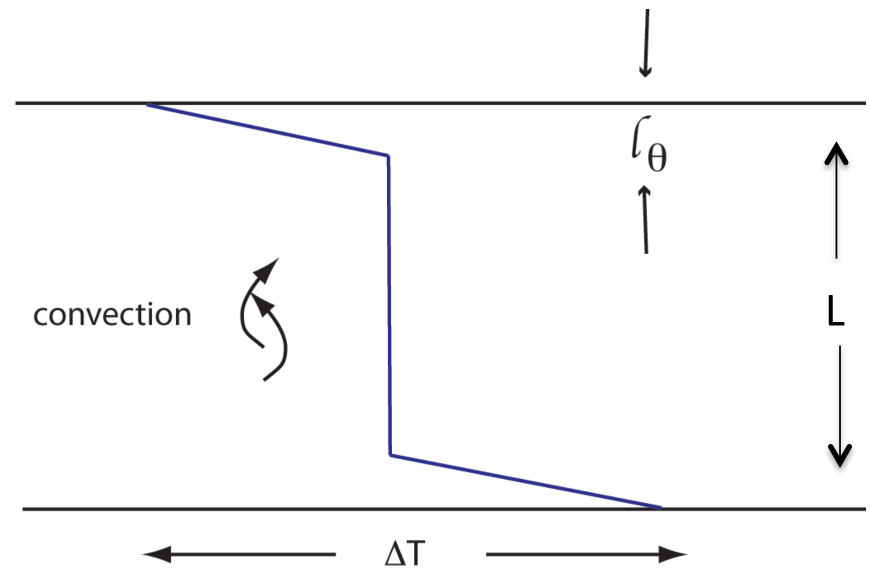
Heat is carried by advection in the interior (e.g. $q_z = \rho$ Cp T v_z). The vertical velocity vanishes at the boundaries, so heat must be carried by conduction across the boundaries (e.g. $q_z = - k\, dT/dz$).

→ The boundary layers are key to understanding convection

Boundary Layer Theory

Heat flow across layer*

$$q_{conv} = \frac{k(\Delta T/2)}{l_\theta}$$



In the initial state (before convection)

$$q_{cond} = \frac{k(\Delta T)}{L}$$

Efficiency of convection $$\frac{q_{conv}}{q_{cond}} = \frac{L}{2l_\theta} = Nu$$ (Nusselt number)

* l_θ is average value

Boundary Layer Instabilities

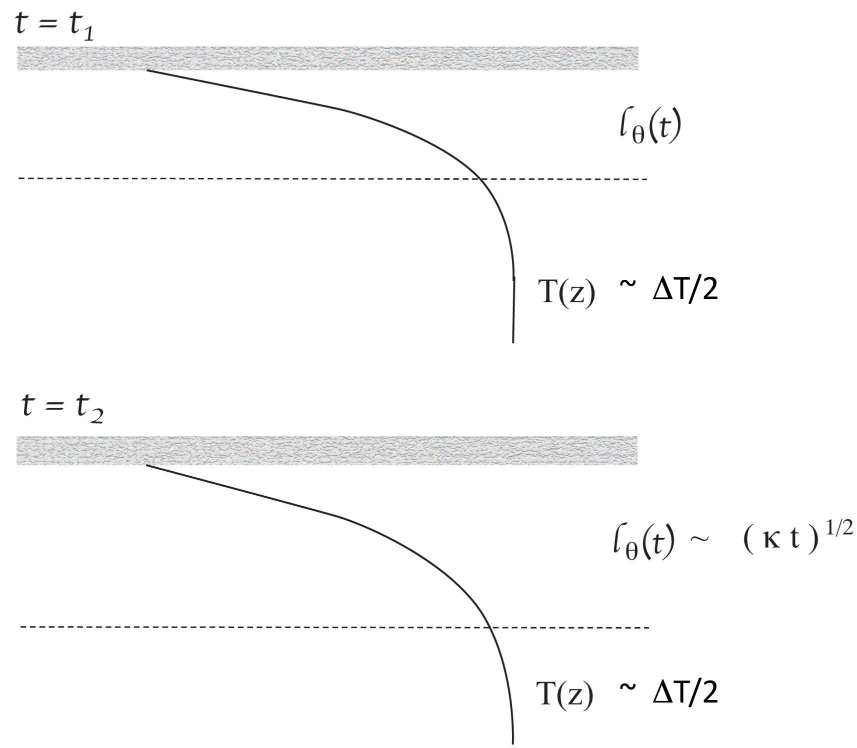
Cold boundary layer grows by conduction into the convecting region

$$l_\theta \approx \sqrt{\kappa t}$$

Eventually the boundary layer becomes unstable at time t_c

Define a local Rayleigh number

$$Ra_l = \frac{\alpha g(\Delta T/2)\, l_\theta^3}{\kappa \nu}$$

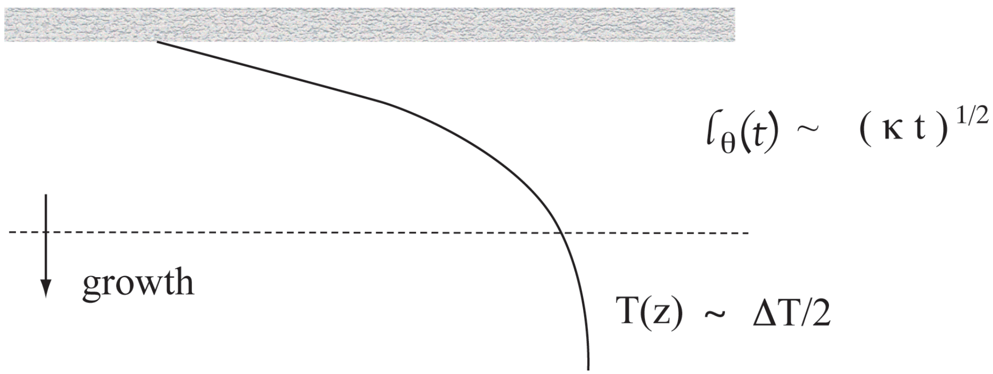


Instability occurs when Ra_l ~ Ra_c ~ 10^3

Average Heat Flow

Heat flow q(t)

$$q(t) = -k\frac{dT}{dz} \approx k\frac{\Delta T/2}{\sqrt{\kappa t}}$$



Time average

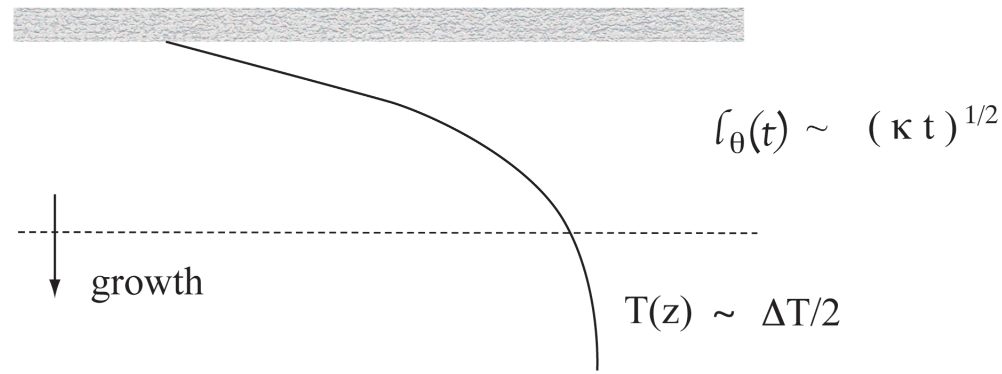
$$\bar{q} \approx \frac{1}{t_c}\int_0^{t_c} k\frac{\Delta T/2}{\sqrt{\kappa t}}\,dt = \frac{k\Delta T}{\sqrt{\kappa t_c}}$$

Recall that $l_\theta^c = \sqrt{\kappa t_c}$ is defined by $Ra_l = Ra_c$

Nu-Ra Relationship

$$\frac{q_{conv}}{q_{cond}} = \frac{L}{2l_\theta} = Nu$$

Time average



$$\bar{q} = k\Delta T / l_\theta^c$$

where

$$Ra_c = \frac{\alpha(\Delta T/2) g \, (l_\theta^c)^3}{\kappa\nu} = \frac{Ra}{2}\left(\frac{l_\theta^c}{L}\right)^3$$

This means that

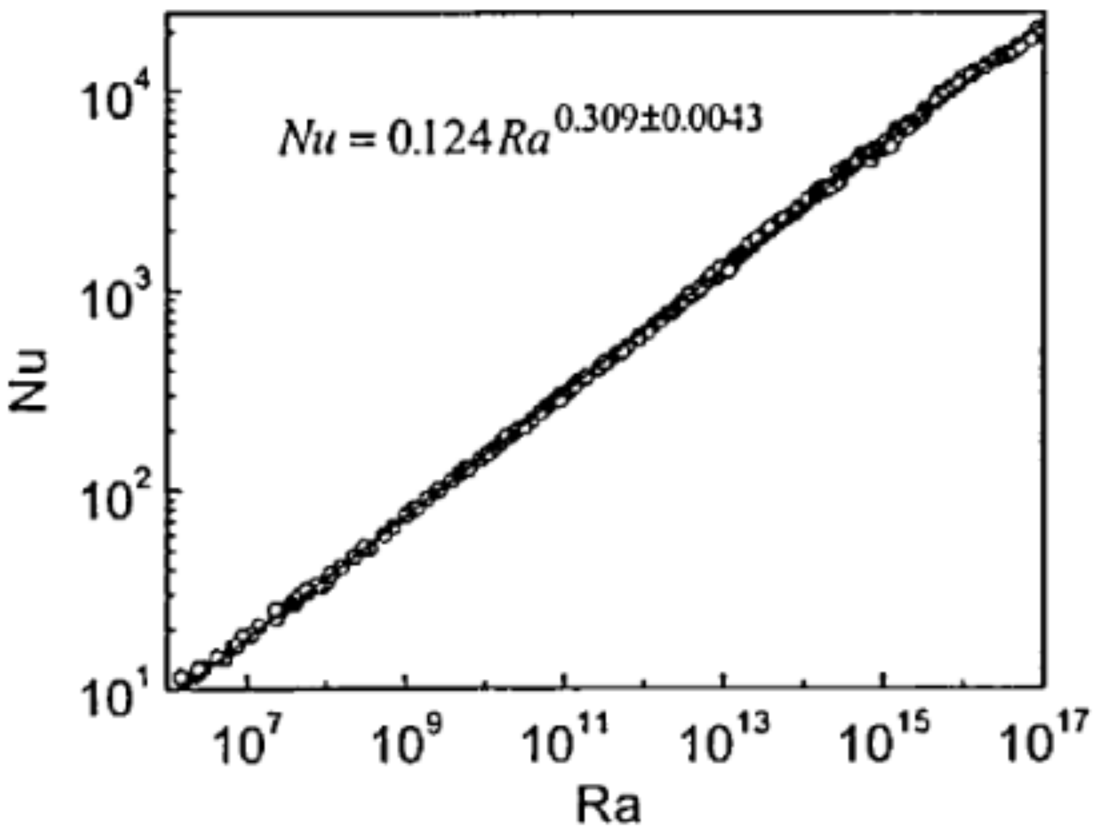
$$\frac{l_\theta^c}{L} = \left(\frac{2Ra_c}{Ra}\right)^{1/3} \qquad \Rightarrow \qquad Nu = \left(\frac{Ra}{2Ra_c}\right)^{1/3}$$

* remember that $l_\theta^c = 2\bar{l}_\theta$

Nu-Ra relationship

Comparison with Experiments

Experiments (Niemela et al. 2000)



Boundary-layer theory

$$Nu = \left(\frac{Ra}{2Ra_c}\right)^{1/3}$$

$$Nu = 0.089 Ra^{0.33}$$

using Kraichnan's estimate for Ra_c ~ 700

Existence of asymptotic regime? Nu ~ $Ra^{1/2}$

Application to Mantle Convection

1. Thickness when layer becomes unstable?

$$Ra_c = \frac{\alpha(\Delta T/2)g\,(l_\theta^c)^3}{\kappa\nu} = \frac{Ra}{2}\left(\frac{l_\theta^c}{L}\right)^3$$

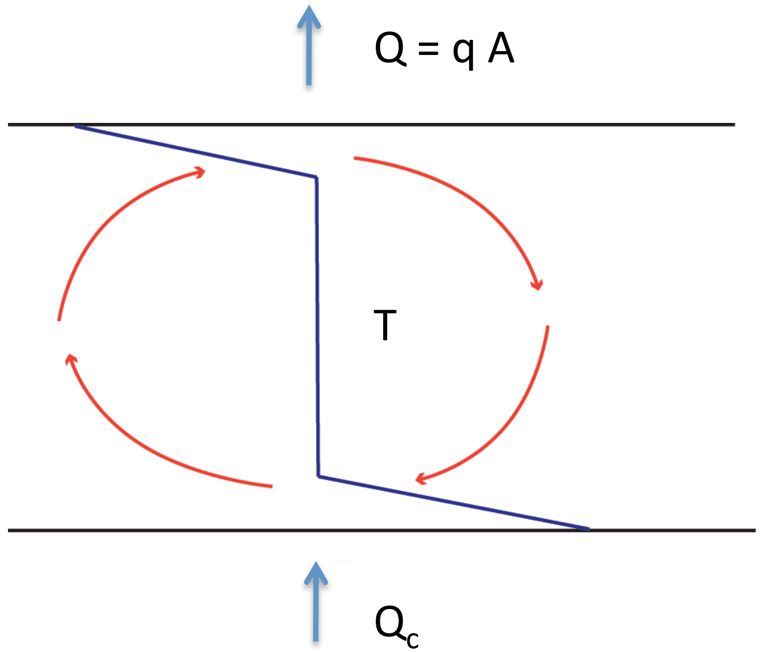
2. Time to become unstable?

$$l_\theta^c = \sqrt{\kappa t_c}$$

3. Dependence of heat flow on layer thickness?

$$\bar{q} = k\Delta T/l_\theta^c$$

5. Thermal Histories



Heat Budget

$$\bar{C}_p M \frac{dT}{dt} = R(t) - Q(t) + Q_c(t)$$

Convection

$$q(t) = \frac{kT(t)}{L} Nu(t) = \frac{kT(t)}{L} \left(\frac{Ra(t)}{2Ra_c} \right)^{1/3}$$

where

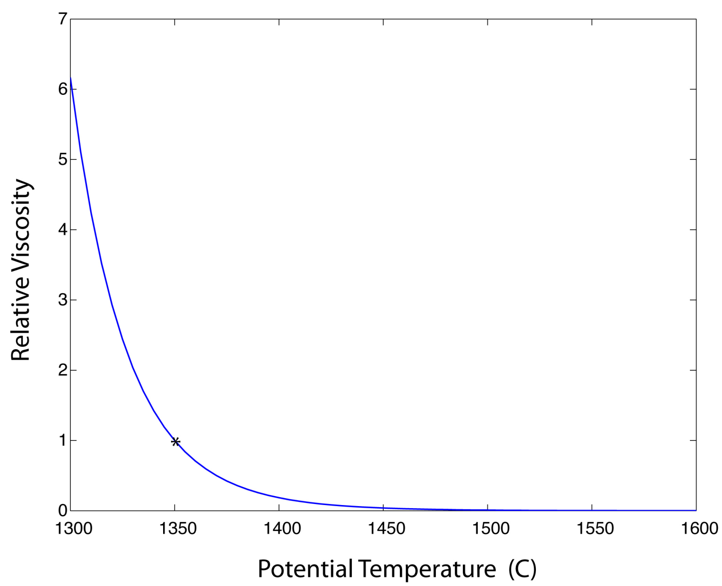
$$Ra(t) = \frac{\alpha g T(t) L^3}{\kappa \nu(t)}$$

Temperature Dependence

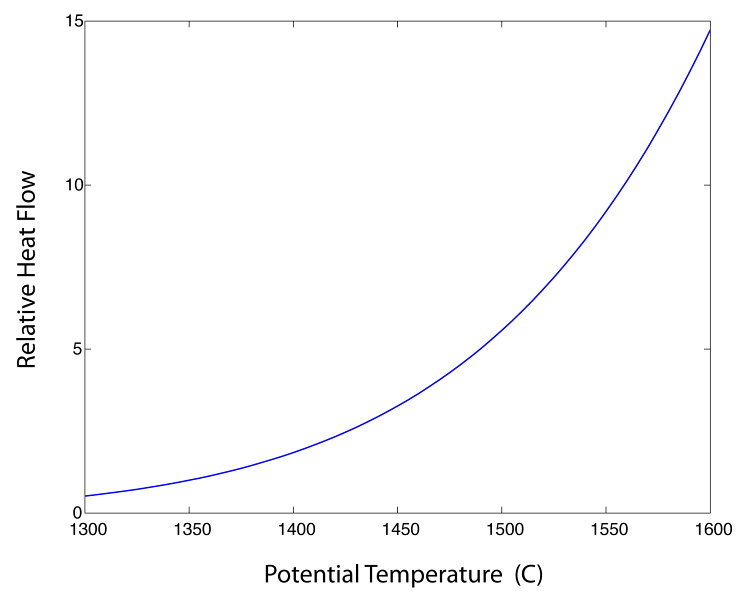
$$\nu(T) \ \propto \ \exp\left(\frac{E}{RT}\right)$$

Change in Heat Flow

Viscosity

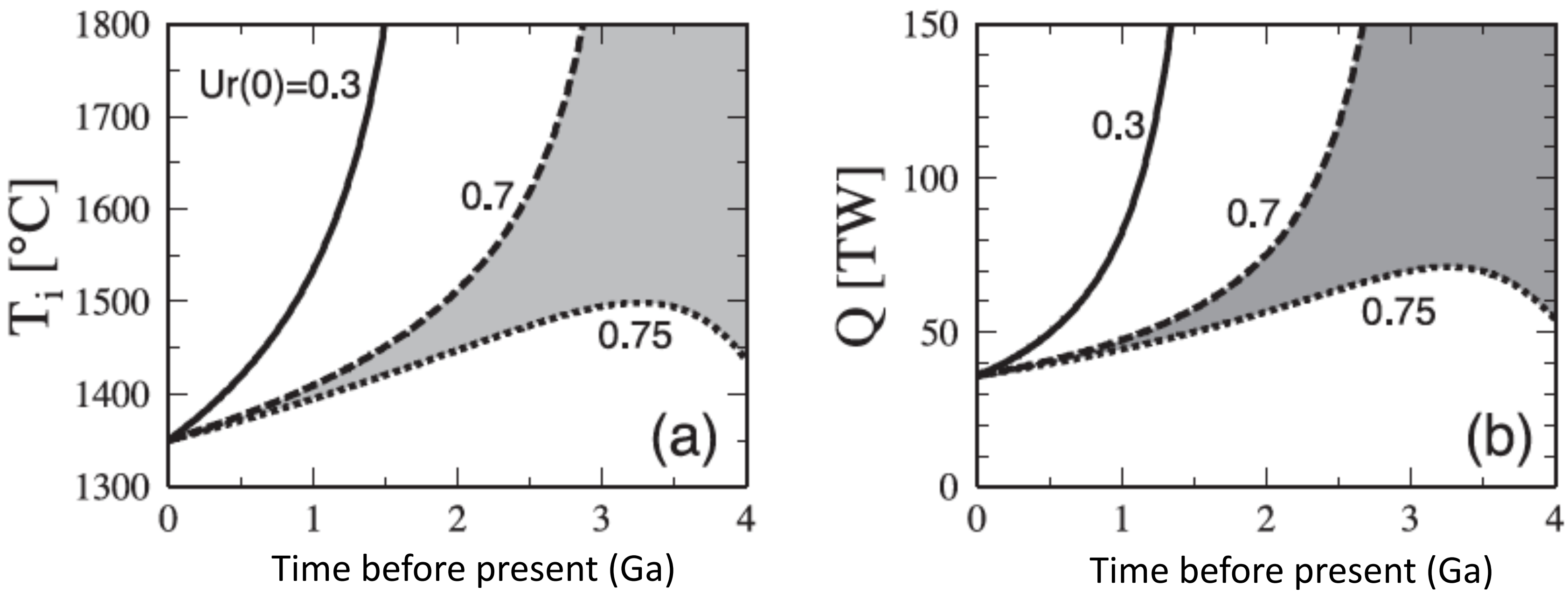


Heat Flow



Strong temperature dependence leads to a thermal "catastrophe" at early times

Thermal Evolution of Mantle



Korenaga (2008)

$$\text{Urey number } Ur = \frac{\text{radiogenic heat}}{\text{convective heat flux}} \approx \frac{11}{36} = 0.3$$

5. Further Complications

Extensions to boundary layer theory?

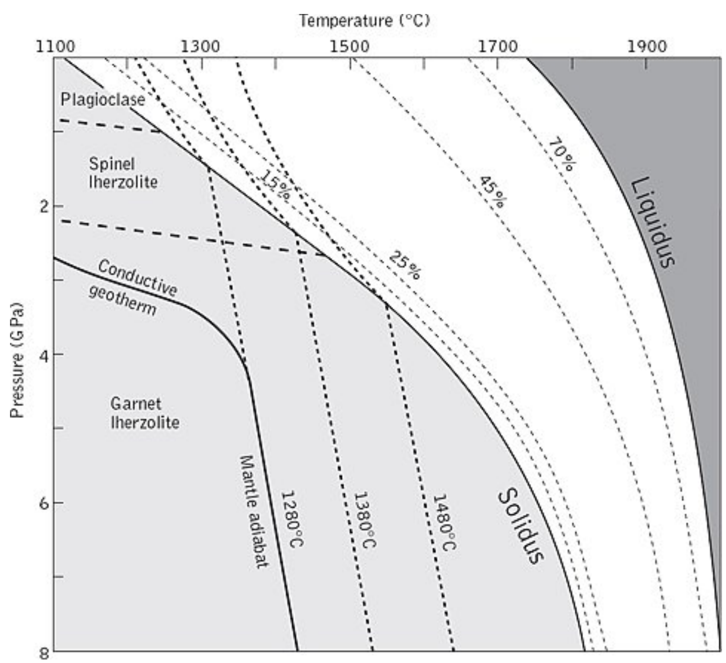
1. Melting and volcanism - affects heat transport and composition (buoyancy)

2. Rheology – some mixture of elastic, viscous and plastic behavior (more Geodynamics 2)

3. Initial conditions?

Melting

Decompression Melting



Melting forms oceanic crust (basalt) and depleted residuum (harzburgite)

More in geochemistry lectures . . .

Densities

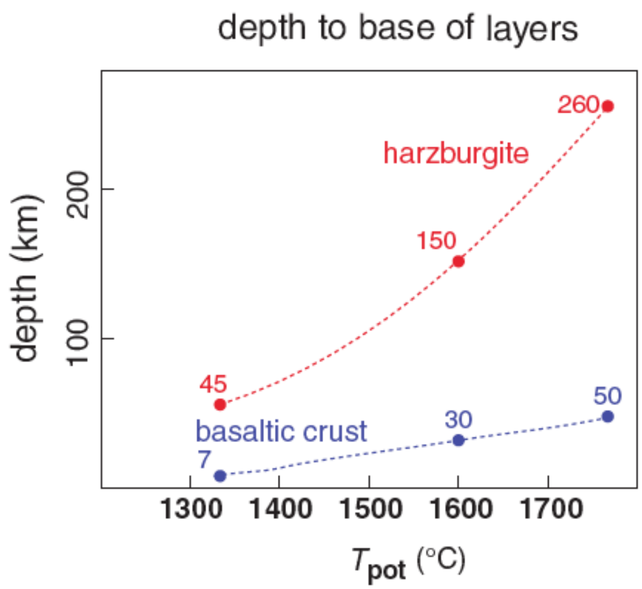
basalt ~ 2.9 g/cm^3

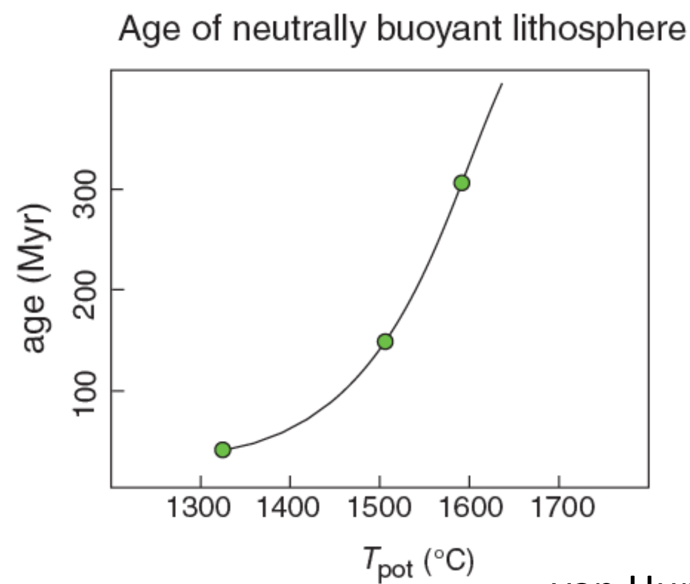
harzburgite ~ 3.2 g/cm^3

lherzolite ~ 3.3 g/cm^3

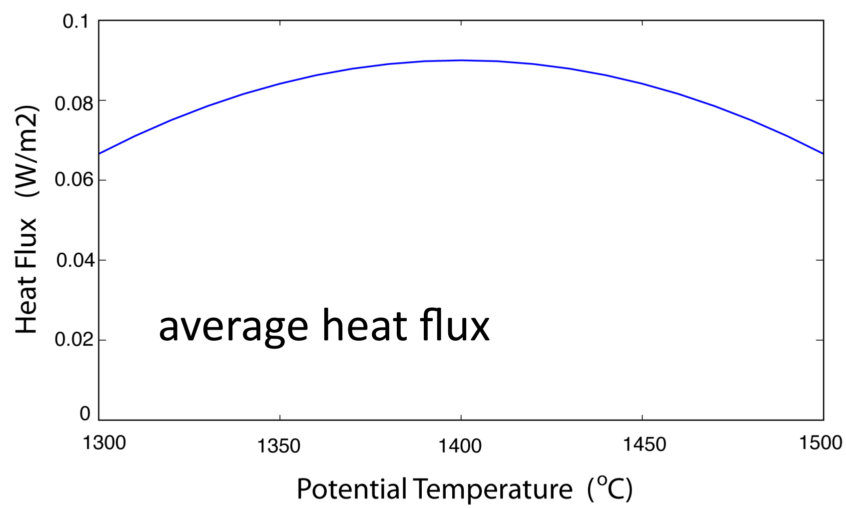
Oxburgh & Parmentier (1977)

Buoyancy of Lithosphere





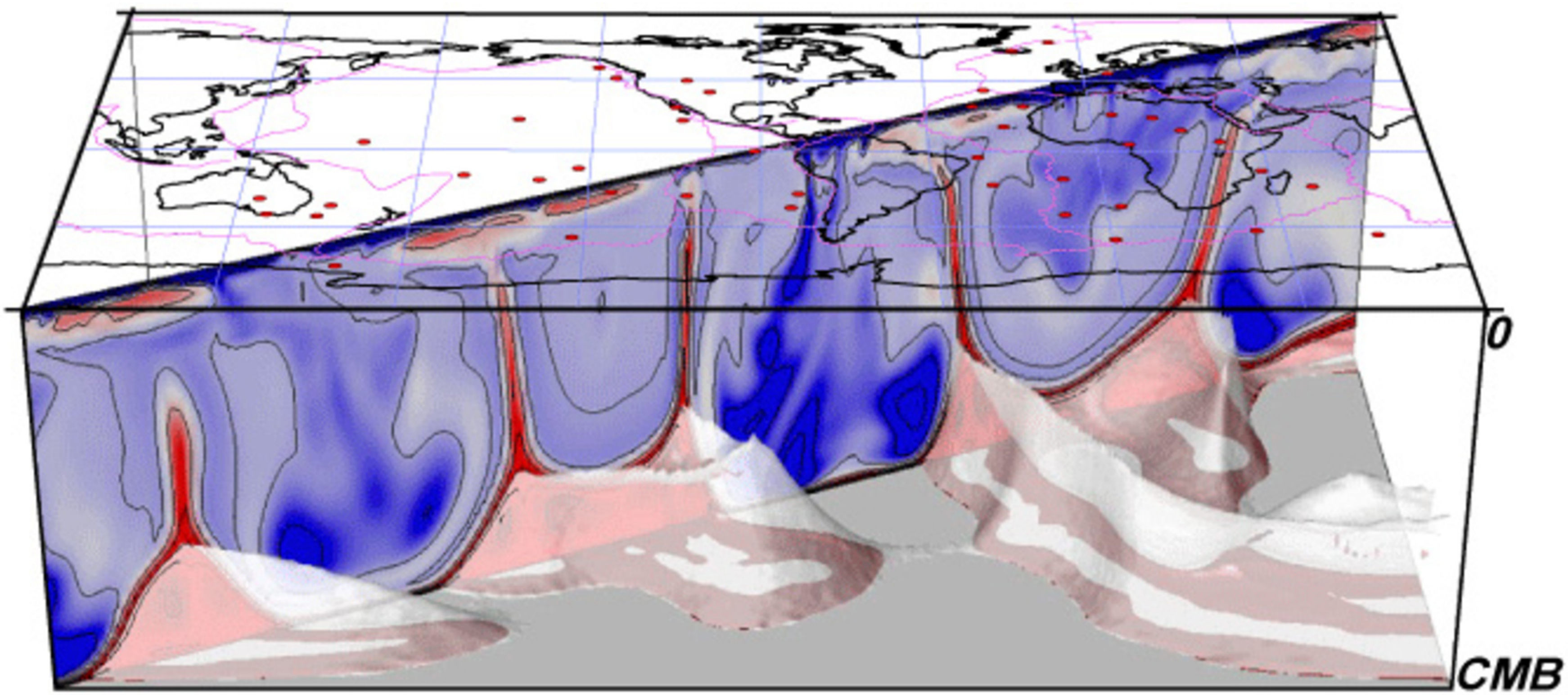
van Hunen et al. (2008)



Sleep (2007)

Not only thermal buoyancy:

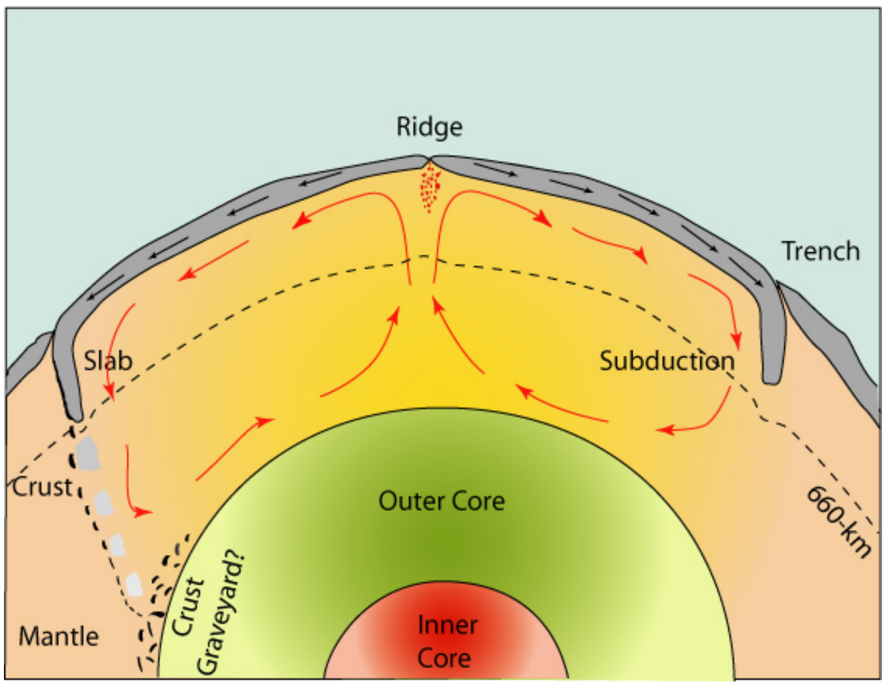
Primordial Layering?



McNamara et al. (2010)

Summary

How good is our theory of mantle convection?



(S. Rost)

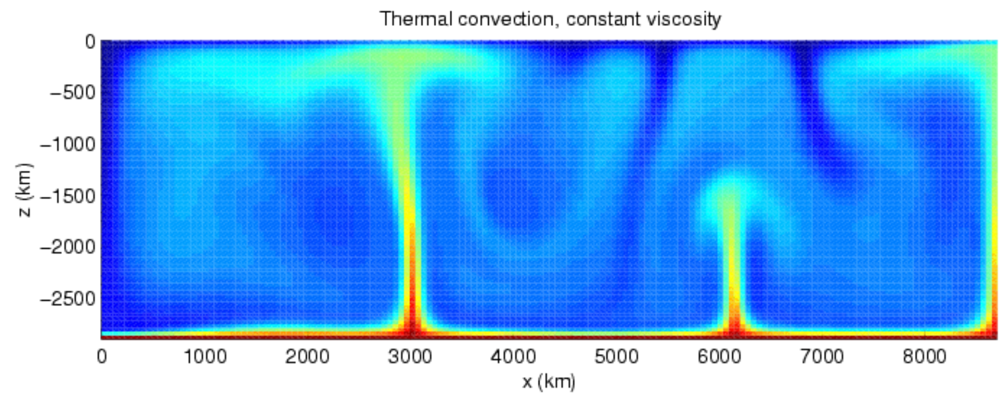
Extrapolation back in time ?

- heat flow?
- number or size of plates?
- continental configuration?
- surface environment/climate?

Geodynamics I: Basics of Thermal Convection

1. Brief overview of governing equations
2. Dimensionless numbers
3. Onset of convection
4. Boundary layer model
5. Scaling relations and thermal evolution

Why?



- Interpret observations (surface, interior) in terms of process and history

Bruce Buffett, UC Berkeley
Presented by Michael Manga, UC Berkeley