Geodynamics I
Basics of Thermal Convection

Quelle & Schmelling (2002)
Other Bodies

Io (NASA)
Wide range of outcomes are predicted by basic equations
Outline

1. A brief overview of the governing equations
2. Introduction to dimensionless numbers
3. Onset of convection (elements of stability theory)
4. The boundary layer model
5. Scaling relations and thermal histories
Some Tools

Fluid Parcel (mass $M$ or volume $V$)

1. Time Derivative (e.g. acceleration $a$)

$$a = \frac{d\mathbf{v}(\mathbf{x}, t)}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial t}$$
Some Tools

Fluid Parcel (mass $M$ or volume $V$)

1. Time Derivative (e.g. acceleration $\mathbf{a}$)

\[
\mathbf{a} = \frac{d \mathbf{v}(x, t)}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}
\]
Some Tools

Fluid Parcel (mass $M$ or volume $V$)

1. Time Derivative (e.g. acceleration $a$)

\[ a = \frac{d \mathbf{v}(x, t)}{dt} = \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = \frac{D}{Dt} \mathbf{v} \]

material derivative
Another Tool

Fluid Parcel (mass $M$ or volume $V$)

2. Reynolds Transport Theorem

$$\frac{d}{dt} \int_{V(t)} f \, dV = \int_{V(t)} \left[ \frac{Df}{Dt} + f(\nabla \cdot \mathbf{v}) \right] \, dV$$
An Example

Mass of parcel

\[ M = \int_{V(t)} \rho \, dV \]

\[ \frac{d}{dt} \int_{V(t)} \rho \, dV = \int_{V(t)} \left[ \frac{D\rho}{Dt} + \rho (\nabla \cdot \mathbf{v}) \right] \, dV = 0 \]
Conservation of Mass

Conservation of mass requires

\[ \frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0 \]

Equivalent form (substitute for \( \frac{D\rho}{Dt} \))

\[ \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho(\nabla \cdot \mathbf{v}) = 0 \]

Incompressible flow

\[ \nabla \cdot \mathbf{v} = 0 \]
Another Example

Conservation of Momentum
\[ \mathbf{p} = \int_{V(t)} \rho \mathbf{v} \, dV \]

Newton’s 2nd Law
\[ \frac{d}{dt} \int_{V(t)} \rho \mathbf{v} \, dV = \mathbf{F} \]

total force on parcel \( V(t) \)

(e.g. gravity, pressure, viscous drag, etc)
Forces on Fluid Parcels

Body & Surface Forces

surface \[ F_s = \int_S \Delta F \, dS \]

body \[ F_b = \int_V \rho g \, dV \]

Cauchy Stress tensor \( T \)

\[ \Delta F = T(x) \cdot n \]

Use divergence theorem

total force \[ F = \int_V \rho g \, dV + \int_S T \cdot n \, dS = \int_V (\rho g + \nabla \cdot T) \, dV \]
Conservation of Momentum

Newton’s 2nd Law

\[
\frac{d}{dt} \int_{V(t)} \rho v \, dV = F = \int_{V(t)} (\rho g + \nabla \cdot T) \, dV
\]

Apply Reynolds Transport

\[
\frac{d}{dt} \int_{V(t)} \rho v \, dV = \int_{V(t)} \left[ \frac{D\rho v}{Dt} + \rho (\nabla \cdot v) \right] \, dV = F
\]

General form of momentum equation (ma = F)

\[
\rho \frac{Dv}{Dt} = \rho g + \nabla \cdot T
\]

How to proceed?
Conservation of Momentum

Newton’s 2nd Law

\[
\frac{d}{dt} \int_{V(t)} \rho \mathbf{v} \, dV = F = \int_{V(t)} (\rho \mathbf{g} + \nabla \cdot \mathbf{T}) \, dV
\]

Apply Reynolds Transport

\[
\frac{d}{dt} \int_{V(t)} \rho \mathbf{v} \, dV = \int_{V(t)} \left[ \rho \frac{D\mathbf{v}}{Dt} \right] \, dV = \int_{V(t)} (\rho \mathbf{g} + \nabla \cdot \mathbf{T}) \, dV
\]

General form of momentum equation \((\mathbf{ma} = F)\)

\[
\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{g} + \nabla \cdot \mathbf{T}
\]

How to proceed?
Constitutive Relations

Perfect fluid (no friction/viscosity)

\[ T = -pI \quad \Rightarrow \quad \nabla \cdot T = -\nabla p \]

General momentum equation becomes Euler’s equation

\[ \rho \frac{Dv}{Dt} = \rho g + \nabla \cdot T \quad \Rightarrow \quad \rho \frac{Dv}{Dt} = \rho g - \nabla p \]

Newtonian Fluid (viscous)

\[ T = -pI + 2 \eta \dot{\varepsilon} \]

Strain-rate tensor \( \dot{\varepsilon} = \frac{1}{2}[\nabla \mathbf{v} + (\nabla \mathbf{v})^T] \)
Constitutive Relations

Perfect fluid (no friction/viscosity)

\[ T = -pI \quad \Rightarrow \quad \nabla \cdot T = -\nabla p \]

General momentum equation becomes Euler’s equation

\[ \rho \frac{Dv}{Dt} = \rho g + \nabla \cdot T \quad \Rightarrow \quad \rho \frac{Dv}{Dt} = \rho g - \nabla p + \eta \nabla^2 v \]

Newtonian Fluid (viscous)

\[ T = -pI + 2\eta \dot{\varepsilon} \quad \text{viscosity} \]

Strain-rate tensor \( \dot{\varepsilon} = \frac{1}{2} [\nabla \mathbf{v} + (\nabla \mathbf{v})^T] \)
Significance of Strain-Rate Tensor

Strain rate of fluid element $L(t)$

$$\Delta L/L(t) = \mathbf{m} \cdot (\dot{\mathbf{e}} \cdot \mathbf{m})$$

Length Stretching

$$\lambda = \lim_{L(0) \to 0} \frac{L(t)}{L(0)}$$

Rate of Stretching

$$\frac{D\lambda}{Dt} = \frac{\dot{L}(t)}{L(0)}$$

$$\frac{1}{\lambda} \frac{D\lambda}{Dt} = \frac{D \ln \lambda}{Dt} = \mathbf{m} \cdot (\dot{\mathbf{e}} \cdot \mathbf{m})$$
Final Conservation Equation

Conservation of Heat* 

\[ H = \int_{V(t)} \rho C_p T \, dV \]

\[ \frac{d}{dt} \int_{V(t)} \rho C_p T \, dV = - \int_{S(t)} \mathbf{q} \cdot \mathbf{n} \, dS + \int_{V(t)} \mathbf{R} \, dV \]

* assumes constant \( \rho \) and \( C_p \)

conduction across surface \( S(t) \)
Final Conservation Equation

Conservation of Heat*

\[ H = \int_{V(t)} \rho C_p T \, dV \]

\[ \frac{d}{dt} \int_{V(t)} \rho C_p T \, dV = - \int_{V(t)} \nabla \cdot \mathbf{q} \, dV + \int_{V(t)} R \, dV \]

* assumes constant \( \rho \) and \( C_p \)
Final Conservation Equation

Conservation of Heat* 

\[ H = \int_{V(t)} \rho C_p T \, dV \]

\[ \int_{V(t)} \left( \rho C_p \frac{DT}{Dt} + \rho C_p T (\nabla \cdot \mathbf{v}) \right) \, dV = \int_{V(t)} (\nabla \cdot k \nabla T + R) \, dV \]

\[ \mathbf{q} = -k \nabla T \]

* assumes constant \( \rho \) and \( C_p \)
Final Conservation Equation

Conservation of Heat* 

\[ H = \int_{V(t)} \rho C_p T \, dV \]

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\int_{V(t)} \left( \rho C_p \frac{DT}{Dt} + \rho C_p T (\nabla \cdot \mathbf{v}) \right) \, dV = \int_{V(t)} (\nabla \cdot k \nabla T + R) \, dV
\]

* assumes constant \(\rho\) and \(C_p\)
Summary for Incompressible Fluid

mass
\[ \frac{D \rho}{Dt} = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{v} = 0 \]

momentum
\[ \rho \frac{D \mathbf{v}}{Dt} = -\nabla p + \eta \nabla^2 \mathbf{v} + \rho \mathbf{g} \]

heat
\[ \rho C_p \frac{D T}{Dt} = k \nabla^2 T + R \]

Subject to boundary conditions on \( \mathbf{v} \) and \( T \). Two useful parameters

\[ \nu = \frac{\eta}{\rho} \approx 10^{18} \text{ m}^2 \text{s}^{-1} \quad \kappa = \frac{k}{\rho C_P} \approx 10^{-6} \text{ m}^2 \text{s}^{-1} \]
Boussinesq Approximation

Equation of state: relate density variations to changes in temperature

\[ \frac{1}{\rho} \frac{\partial \rho}{\partial T} = -\alpha \]

thermal expansion
Dimensionless Numbers

Why?!

We often describe quantities in SI units (m, kg, s, etc.)

Other choices
- distance L in terms of R
- velocity V in terms of \( v_{\text{max}} \)
- time t in terms of \( R/v_{\text{max}} \)

Order of Magnitudes:
\[
\nabla v \sim \frac{v_{\text{max}} - 0}{R} \sim \mathcal{O}(1)
\]
\[
\nabla^2 v \sim \frac{v_{\text{max}}/R - 0}{R} \sim \frac{v_{\text{max}}}{R^2} \sim \mathcal{O}(1)
\]
Dimensionless Numbers

Define dimensionless variables

\[ x' = x / R \]
\[ v' = v / v_{\text{max}} \]
\[ t' = t / \tau \]

Navier-Stokes (in SI units)

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla P + \eta \nabla^2 \mathbf{v} \]

Change variables to \( x', t', v' \)

\[ \left( \frac{\partial v'}{\partial t'} + v' \cdot \nabla v' \right) = -\nabla P' + \frac{1}{Re} \nabla^2 v' \]

\[ Re = \frac{v_{\text{max}} R}{(\eta / \rho)} = \frac{v_{\text{max}} R}{\nu} \]

Reynolds number
Convection Problem

Dimensionless variables
\[ x' = x / L \]
\[ T' = T / \Delta T \]
\[ t' = t / \tau \quad \text{where} \quad \tau = L^2 / \kappa \]
\[ v' = v / (L/\tau) \]

Change variables to dimensionless quantities

1. \[ \nabla \cdot v' = 0 \]

2. \[ \frac{1}{Pr} \left( \frac{\partial v'}{\partial t'} + v' \cdot \nabla v' \right) = -\nabla P' + Ra T' \hat{z} + \nabla^2 v' \]

3. \[ \frac{\partial T'}{\partial t'} + v' \cdot \nabla T' = \nabla^2 T' + R' \]

Dimensionless numbers

[Diagram showing a convective flow with dimensionless variables and equations]

Dimensionless numbers

\[ Pr = \frac{\nu}{\kappa} \]

\[ Ra = \frac{\alpha g \Delta T L^3}{\kappa \nu} \]
Convection Problem

Dimensionless variables

\[ x' = \frac{x}{L} \]
\[ T' = \frac{T}{\Delta T} \]
\[ t' = \frac{t}{\tau} \quad \text{where } \tau = \frac{L^2}{\kappa} \]
\[ v' = \frac{v}{(L/\tau)} \]

Change variables to dimensionless quantities

1. \[ \nabla \cdot v' = 0 \]

2. \[ \frac{1}{Pr} \left( \frac{\partial v'}{\partial t'} + v' \cdot \nabla v' \right) = -\nabla P' + Ra T' \hat{z} + \nabla^2 v' \]

3. \[ \frac{\partial T'}{\partial t'} + v' \cdot \nabla T' = \nabla^2 T' + R' \]

Dimensionless numbers

\[ Pr = \frac{\nu}{\kappa} \sim 10^{24} \]  
(mantle)

\[ Ra = \frac{\alpha g \Delta T L^3}{\kappa \nu} \]
Interpretation of Rayleigh Number

Velocity of Parcel \( v \approx \Delta \rho g L^2/\eta \)

For hot fluid \( |\Delta \rho| = \rho \alpha \Delta T \)

Advection time \( \tau_a = \frac{L}{v} \)

How does this compare to conduction time \( (\tau_c = L^2/\kappa) \)?

\[
\frac{\tau_c}{\tau_a} = \frac{vL}{\kappa} = \frac{\rho \alpha g \Delta T L^3}{\kappa \eta}
\]

(Rayleigh number)

For Earth \( L = 2900 \text{ km}, \ \Delta T \sim 3000 \text{ K}, \ \text{Ra} \sim 10^8 \) (critical \( \text{Ra}_c = 10^3 \))
Onset of Convection

When does convection begin?

Consider time evolution of a small perturbation in an initially conductive state

\[ T(x, y, z, t) = T_0(z) + \delta T(x, y, z) e^{\sigma t} \]

\[ \mathbf{v}(x, y, z, t) = \delta \mathbf{v}(x, y, z) e^{\sigma t} \]

Substitute into (linearized) equations and solve for growth rate \( \sigma \)
Onset of Convection

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Substitute into (linearized) equations and solve for growth rate \( \sigma \)
Heat is carried by advection in the interior (e.g. \( q_z = \rho C_p T v_z \)). The vertical velocity vanishes at the boundaries, so heat must be carried by conduction across the boundaries (e.g. \( q_z = -k \frac{dT}{dz} \)).

The boundary layers are key to understanding convection.
Boundary Layer Theory

Heat flow across layer*

\[ q_{\text{conv}} = \frac{k(\Delta T/2)}{l_\theta} \]

In the initial state (before convection)

\[ q_{\text{cond}} = \frac{k(\Delta T)}{L} \]

Efficiency of convection

\[ \frac{q_{\text{conv}}}{q_{\text{cond}}} = \frac{L}{2l_\theta} = Nu \]

(Nusselt number)

* \( l_\theta \) is average value
Boundary Layer Instabilities

Cold boundary layer grows by conduction into the convecting region

\[ l_\theta \approx \sqrt{\kappa t} \]

Eventually the boundary layer becomes unstable at time \( t_c \)

Define a local Rayleigh number

\[ Ra_l = \frac{\alpha g (\Delta T / 2)}{\kappa \nu} l_\theta^3 \]

Instability occurs when \( Ra_l \sim Ra_c \sim 10^3 \)
Average Heat Flow

Heat flow $q(t)$

$$q(t) = -k \frac{dT}{dz} \approx k \frac{\Delta T/2}{\sqrt{\kappa t}}$$

Time average

$$\bar{q} \approx \frac{1}{t_c} \int_0^{t_c} k \frac{\Delta T/2}{\sqrt{\kappa t}} dt = \frac{k \Delta T}{\sqrt{\kappa t_c}}$$

Recall that $l_\theta^c = \sqrt{\kappa t_c}$ is defined by $Ra_l = Ra_c$
Nu-Ra Relationship

Time average

\[ \bar{q} = k \Delta T / l_\theta^c \]

where

\[ Ra_c = \frac{\alpha(\Delta T/2)g(l_\theta^c)^3}{\kappa \nu} = \frac{Ra}{2} \left( \frac{l_\theta^c}{L} \right)^3 \]

This means that

\[ \frac{l_\theta^c}{L} = \left( \frac{2Ra_c}{Ra} \right)^{1/3} \]

\[ Nu = \frac{L}{2l_\theta} = \left( \frac{Ra}{2Ra_c} \right)^{1/3} \]

* remember that \( l_\theta^c = 2l_\theta \)

Nu-Ra relationship
Comparison with Experiments

Experiments (Niemela et al. 2000)

\[ Nu = 0.124 Ra^{0.309 \pm 0.0043} \]

Boundary-layer theory

\[ Nu = \left( \frac{Ra}{2 Ra_c} \right)^{1/3} \]

\[ Nu = 0.089 Ra^{0.33} \]

using Kraichnan’s estimate for \( Ra_c \sim 700 \)

Existence of asymptotic regime? \( Nu \sim Ra^{1/2} \)
Application to Mantle Convection

1. Thickness of lithospheric plates

\[ \frac{l_\theta^c}{L} = \left( \frac{2Ra_c}{Ra} \right)^{1/3} \]

for \( Ra = 10^8, Ra_c = 10^3, L = 2900 \text{ km} \) we get \( l_\theta = 80 \text{ km} \)

2. Velocity of lithosphere

Cooling time

\[ t_c = \frac{l_\theta^2}{\kappa} \]

Velocity

\[ V = \frac{L}{t_c} = \left( \frac{Ra}{2Ra_c} \right)^{2/3} \frac{\kappa}{L} \]

\( \sim 1.5 \text{ cm/year} \)
Application to Mantle Convection

3. Heat Flow

\[ q = \left( \frac{k \Delta T}{L} \right) N_u \quad \text{where} \quad N_u = \left( \frac{Ra}{2Ra_c} \right)^{1/3} \]

for \( Ra = 10^8, Ra_c = 10^3, L = 2900 \text{ km} \) we get \( Nu \sim 36 \)

Taking \( k = 3 \text{ W m}^{-1} \text{K}^{-1}, \Delta T = 3000 \text{ K} \) yields \( q_{\text{cond}} = 3 \text{ mW m}^{-2} \) or \( Q_{\text{cond}} = 1.4 \text{ TW} \)

\( Q_{\text{conv}} \sim 50 \text{ TW} \) (instead of 46 TW)
Thermal Histories

Heat Budget

\[ \bar{C}_p M \frac{dT}{dt} = R(t) - Q(t) + Q_c(t) \]

Parameterized Convection

\[ q(t) = \frac{kT(t)}{L} \tilde{N}_u(t) = \frac{kT(t)}{L} \left( \frac{Ra(t)}{2Ra_c} \right)^{1/3} \]

where

\[ Ra(t) = \frac{\alpha g T(t) L^3}{\kappa \nu(t)} \]

Temperature Dependence

\[ \nu(T) \propto \exp \left( \frac{E}{RT} \right) \]
Further Complications

Extensions to boundary layer theory?

1. Melting and volcanism - affects heat transport and composition (buoyancy)
2. Rheology – some mixture of elastic, viscous and plastic behavior
3. Initial condition?
Melting

Decompression Melting

Melting forms oceanic crust (basalt) and depleted residuum (harzburgite)

Densities

- basalt $\sim 2.9 \text{ g/cm}^3$
- harzburgite $\sim 3.2 \text{ g/cm}^3$
- lherzolite $\sim 3.3 \text{ g/cm}^3$

Oxburgh & Parmentier (1977)
Buoyancy of Lithosphere

van Hunen et al. (2008)

Sleep (2007)
Rheology

Surface Force \[ \mathbf{T} = -p \mathbf{I} + \tau \] where \( \tau \) is deviatoric stress tensor

Newtonian fluid \[ \tau = 2 \eta \dot{\varepsilon} \] where strain rate \( \dot{\varepsilon} = \frac{1}{2} (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) \)

More general flow laws

\[ \dot{\varepsilon} = f(T, P, \tau, \ldots) \tau \]

Other variables: grain size, volatiles, partial melt, deformation history
Primordial Layering?

McNamara et al. (2010)
How good is our theory of mantle convection?

Extrapolation back in time?

- heat flow?
- number or size of plates?
- continental configuration?
- surface environment/climate?

(S. Rost)