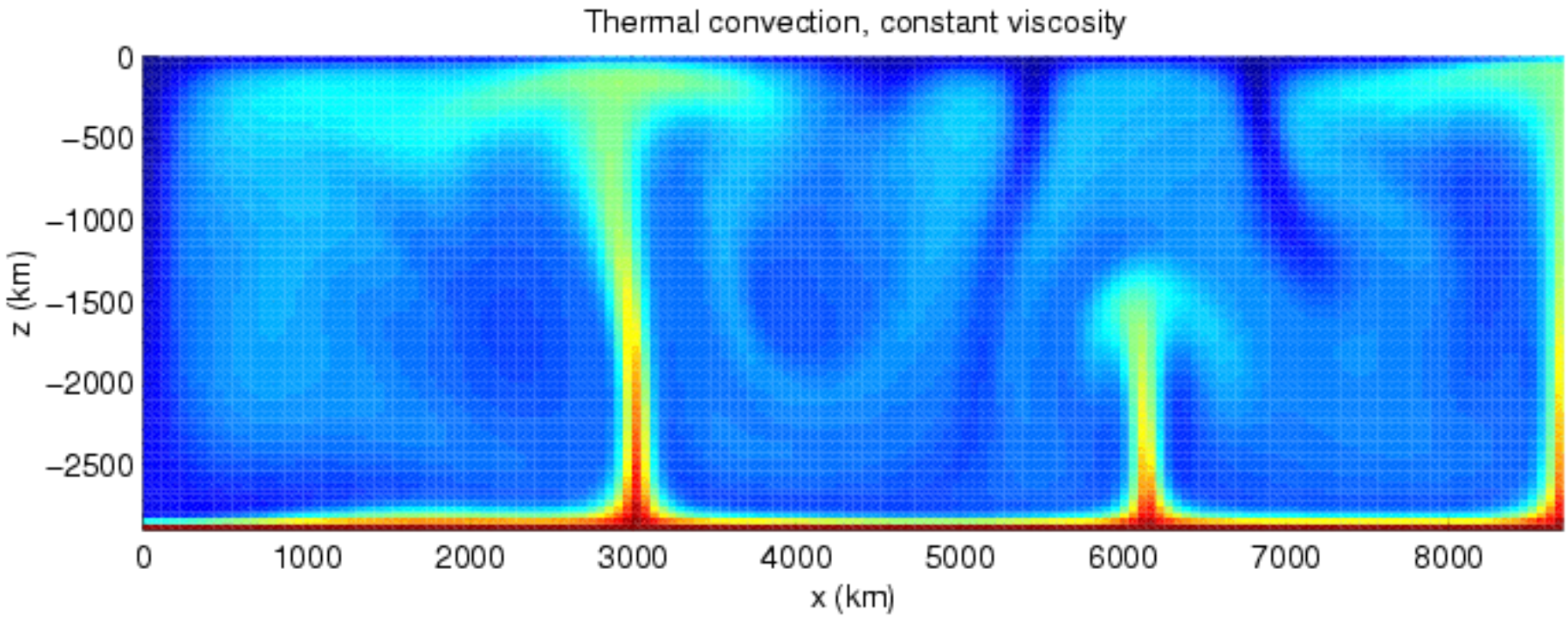
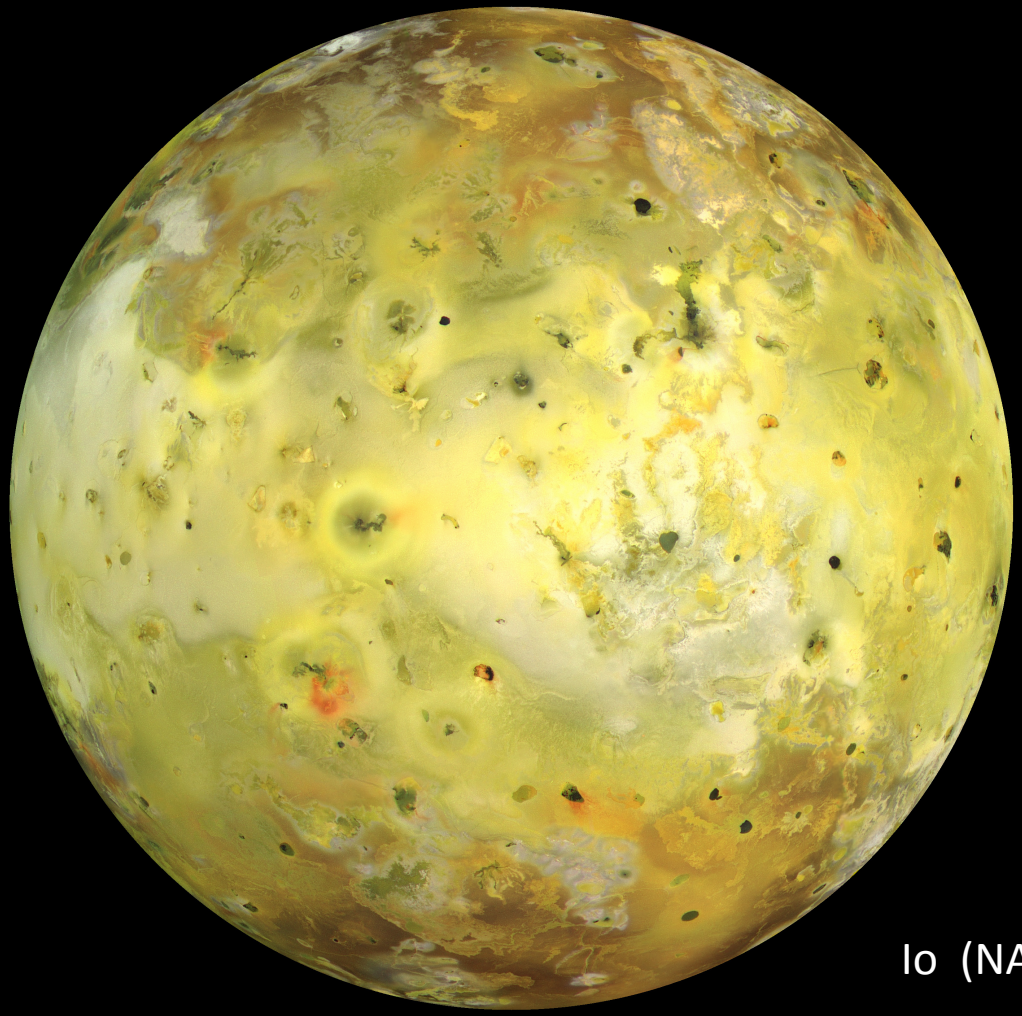
Geodynamics I
Basics of Thermal Convection



Quelle & Schmelling (2002)

Other Bodies

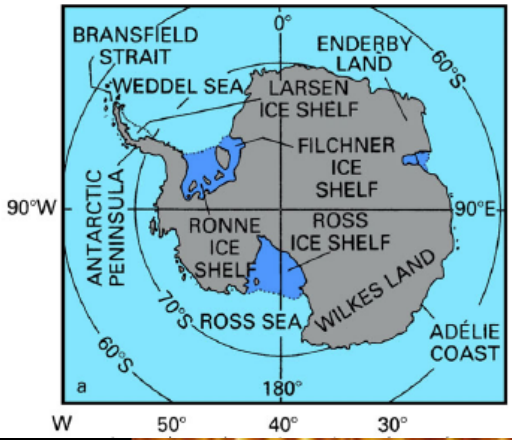


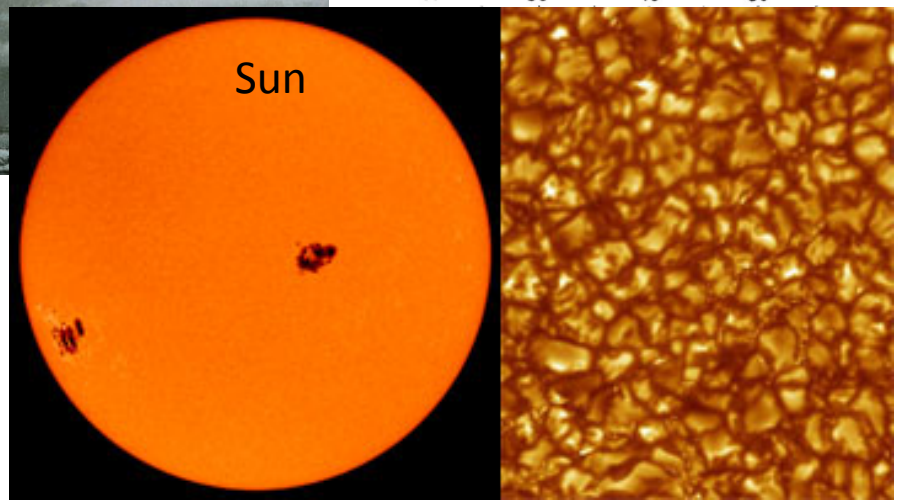
Io (NASA)

Other Contexts

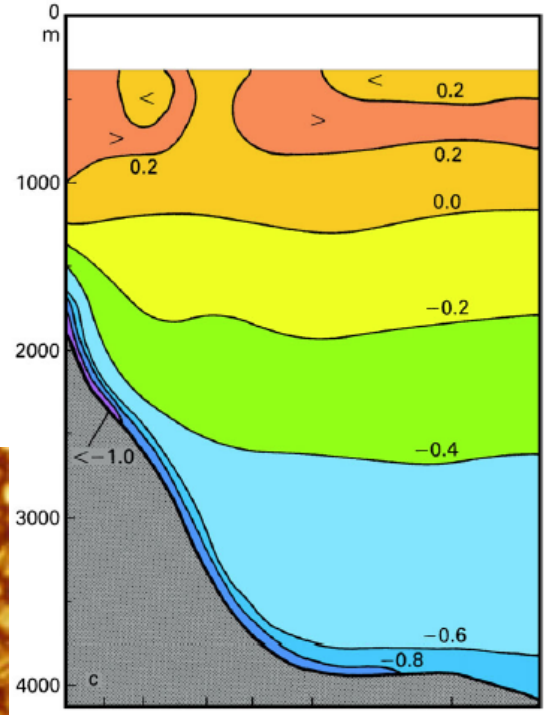
Atmosphere

NOAA





Ocean



Rahmstorf 2006

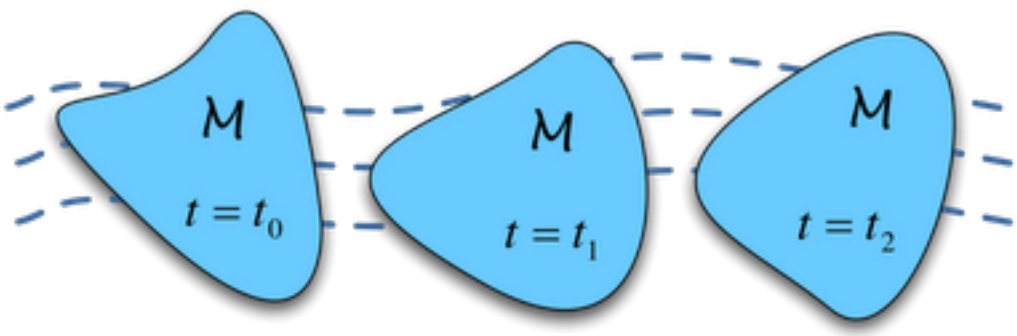
Wide range of outcomes are predicted by basic equations

Outline

1. A brief overview of the governing equations
2. Introduction to dimensionless numbers
3. Onset of convection (elements of stability theory)
4. The boundary layer model
5. Scaling relations and thermal histories

Some Tools

Fluid Parcel (mass M or volume V)



1. Time Derivative (e.g. acceleration **a**)

$$\mathbf{a} = \frac{d\,\mathbf{v}(\mathbf{x}, t)}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial t}$$

Some Tools

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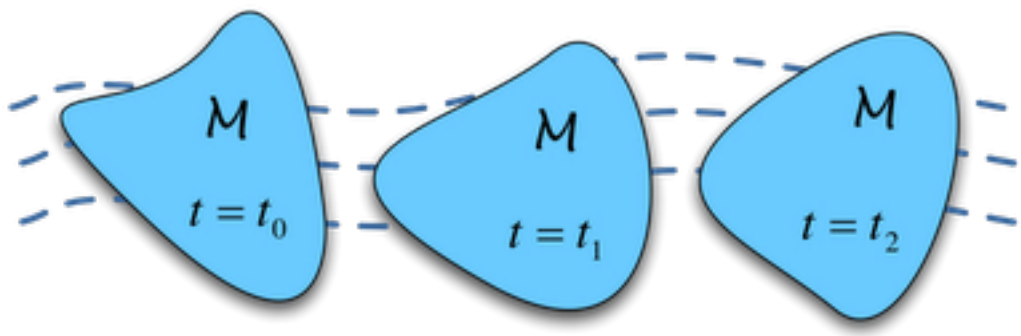

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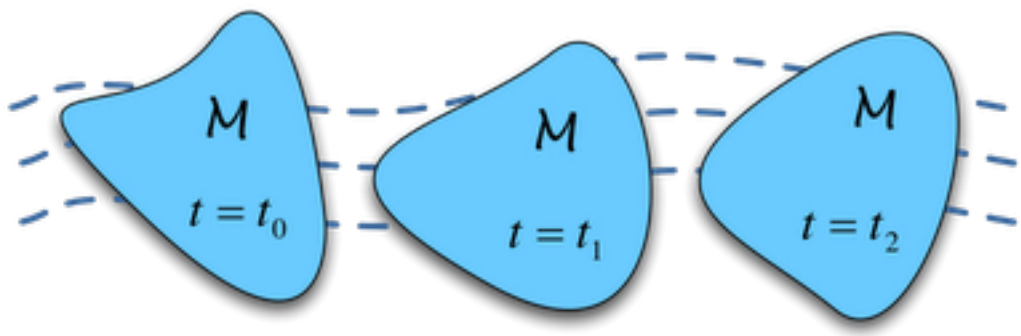


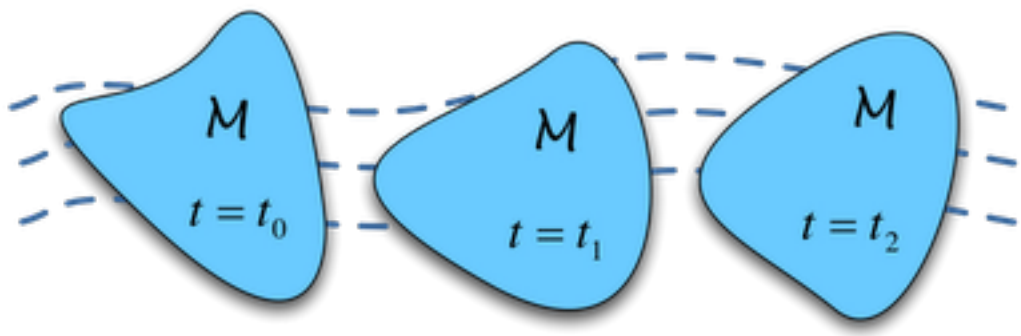
1. Time Derivative (e.g acceleration **a**)

$$\mathbf{a} = \frac{d\,\mathbf{v}(\mathbf{x},t)}{dt} = \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} = \frac{D}{Dt}\mathbf{v}$$

material derivative

Another Tool

Fluid Parcel (mass M or volume V)

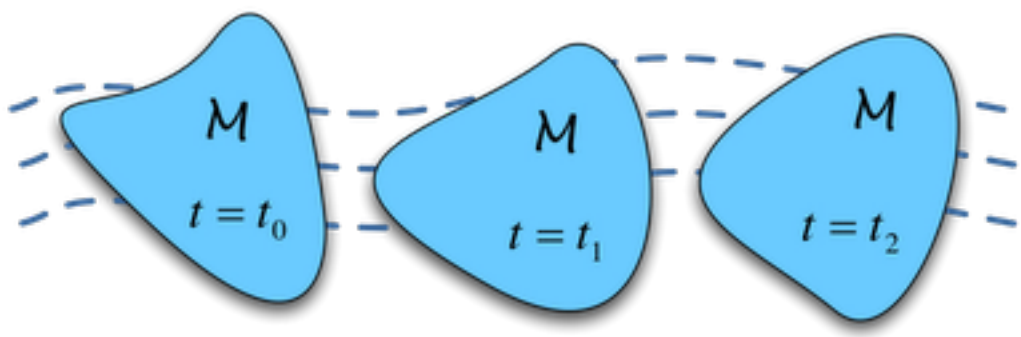


2. Reynolds Transport Theorem

$$\frac{d}{dt}\int_{V(t)} f\,dV = \int_{V(t)} \left[\frac{Df}{Dt} + f(\nabla\cdot\mathbf{v})\right] dV$$

An Example

Mass of parcel $$M = \int_{V(t)} \rho \, dV$$



$$\frac{d}{dt} \int_{V(t)} \rho \, dV = \int_{V(t)} \left[\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) \right] dV = 0$$

Conservation of Mass

Conservation of mass requires

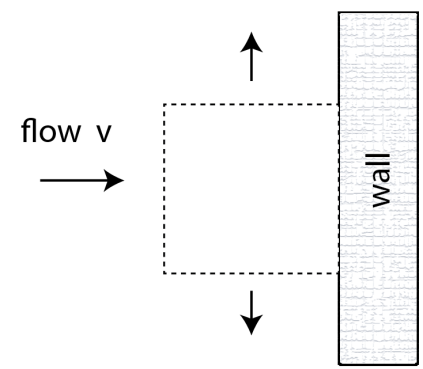
$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0$$

Equivalent form (substitute for Dρ/Dt)

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho(\nabla \cdot \mathbf{v}) = 0$$

Incompressible flow

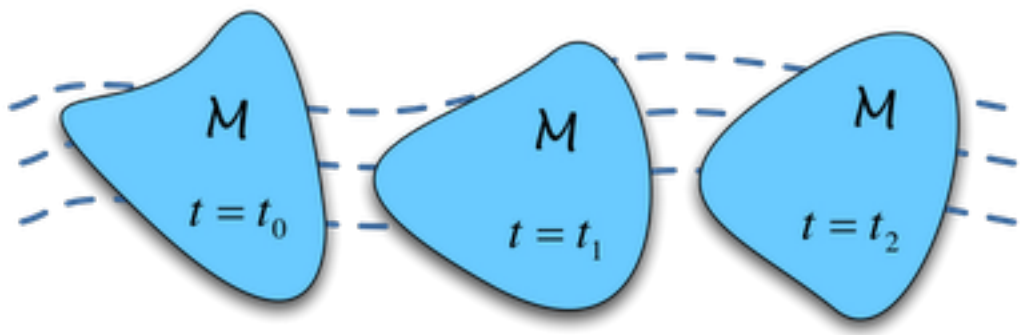
$$\nabla \cdot \mathbf{v} = 0$$



Another Example

Conservation of Momentum

$$\mathbf{p} = \int_{V(t)} \rho \mathbf{v} \, dV$$



Newton's 2nd Law

$$\frac{d}{dt} \int_{V(t)} \rho \mathbf{v} \, dV = \mathbf{F}$$

total force on parcel V(t)

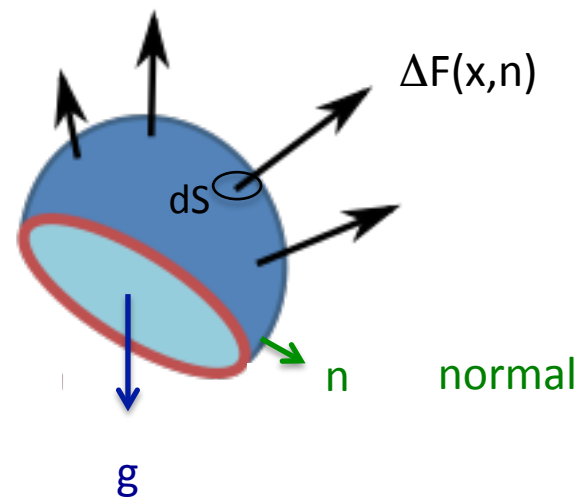
(e.g. gravity, pressure, viscous drag, etc)

Forces on Fluid Parcels

Body & Surface Forces

surface $F_s = \int_S \Delta F \, dS$

body $F_b = \int_V \rho g \, dV$



Cauchy Stress tensor T

$$\Delta F = T(x) \cdot n$$

Use divergence theorem

total force $F = \int_V \rho g \, dV + \int_S T \cdot n \, dS = \int_V (\rho g + \nabla \cdot T) \, dV$

Conservation of Momentum

Newton's 2nd Law

$$\frac{d}{dt}\int_{V(t)} \rho\mathbf{v}\, dV = \mathbf{F} = \int_{V(t)} (\rho\mathbf{g} + \nabla\cdot\mathbf{T})\, dV$$

Apply Reynolds Transport

$$\frac{d}{dt}\int_{V(t)} \rho\mathbf{v}\, dV = \int_{V(t)} \left[\frac{D\rho\mathbf{v}}{Dt} + \rho\mathbf{v}(\nabla\cdot\mathbf{v})\right] dV = \mathbf{F}$$

General form of momentum equation (ma = F)

$$\rho\frac{D\mathbf{v}}{Dt} = \rho\mathbf{g} + \nabla\cdot\mathbf{T}$$

How to proceed?

Conservation of Momentum

Newton's 2nd Law

$$\frac{d}{dt}\int_{V(t)} \rho \mathbf{v}\, dV = \mathbf{F} = \int_{V(t)} (\rho \mathbf{g} + \nabla \cdot \mathbf{T})\, dV$$

Apply Reynolds Transport

$$\frac{d}{dt}\int_{V(t)} \rho \mathbf{v}\, dV = \int_{V(t)} \left[\rho \frac{D\mathbf{v}}{Dt}\right] dV = \int_{V(t)} (\rho \mathbf{g} + \nabla \cdot \mathbf{T})\, dV$$

General form of momentum equation (ma = F)

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{g} + \nabla \cdot \mathbf{T}$$

How to proceed?

Constitutive Relations

Perfect fluid (no friction/viscosity)

$$T = -pI \quad \Rightarrow \quad \nabla \cdot T = -\nabla p$$

General momentum equation becomes Euler's equation

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{g} + \nabla \cdot \mathbf{T} \quad \Rightarrow \quad \rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{g} - \nabla p$$

Newtonian Fluid (viscous)

$$T = -pI + 2\eta \dot{e}$$

viscosity

Strain-rate tensor $\dot{e} = \frac{1}{2}[\nabla \mathbf{v} + (\nabla \mathbf{v})^T]$

Constitutive Relations

Perfect fluid (no friction/viscosity)

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Navier - Stokes

General momentum equation becomes Euler's equation

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{g} + \nabla \cdot \mathbf{T} \quad \Rightarrow \quad \rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{g} - \nabla p + \eta \nabla^2 \mathbf{v}$$

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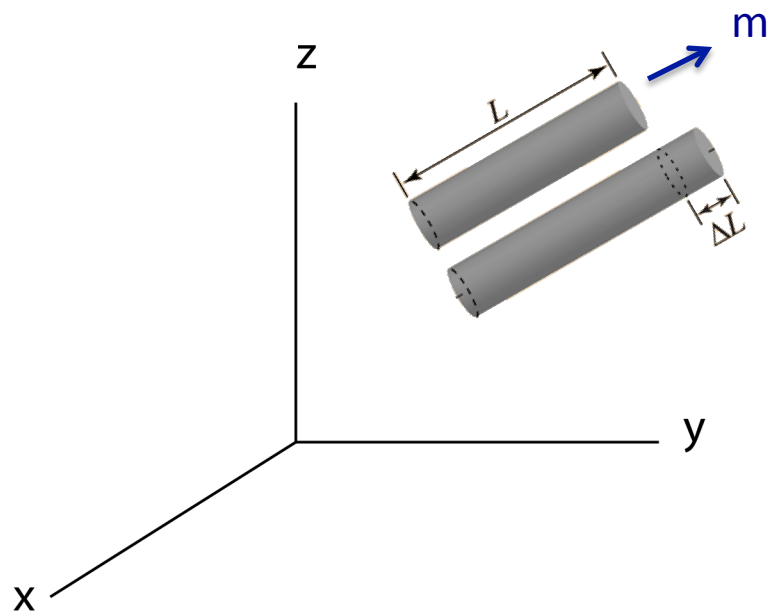
viscosity

Strain-rate tensor $\dot{e} = \frac{1}{2}[\nabla \mathbf{v} + (\nabla \mathbf{v})^T]$

Significance of Strain-Rate Tensor

Strain rate of fluid element L(t)

$$\dot{\Delta} L/L(t) = \mathbf{m} \cdot (\dot{\mathbf{e}} \cdot \mathbf{m})$$



Length Stretching

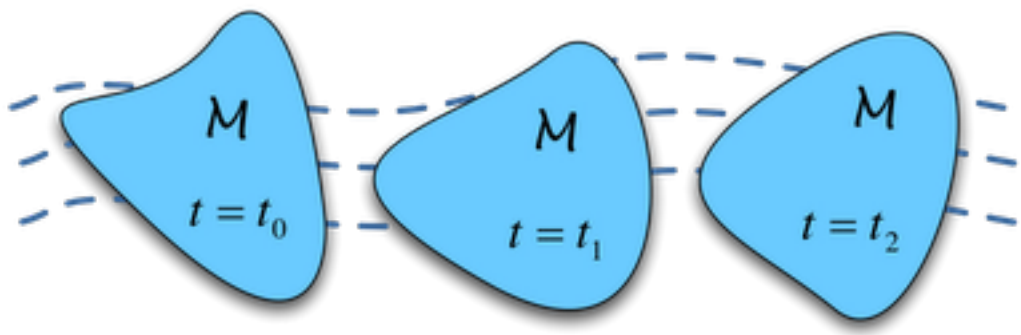
$$\lambda = \lim_{L(0) \to 0} L(t)/L(0)$$

Rate of Stretching

$$\frac{D\lambda}{Dt} = \frac{\dot{L}(t)}{L(0)} \quad \Rightarrow \quad \frac{1}{\lambda}\frac{D\lambda}{Dt} = \frac{D \ln \lambda}{Dt} = \mathbf{m} \cdot (\dot{\mathbf{e}} \cdot \mathbf{m})$$

Final Conservation Equation

Conservation of Heat* $$H = \int_{V(t)} \rho C_p T \, dV$$



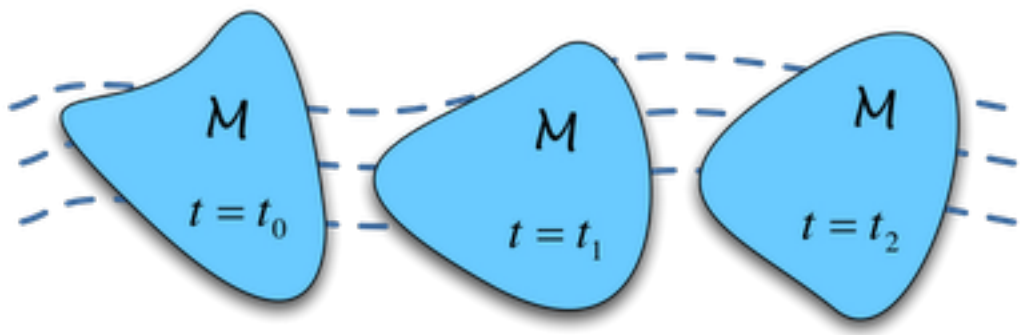
$$\frac{d}{dt} \int_{V(t)} \rho C_p T \, dV = - \int_{S(t)} \mathbf{q} \cdot \mathbf{n} \, dS + \int_{V(t)} R \, dV$$

conduction across surface S(t)

* assumes constant ρ and C_p

Final Conservation Equation

Conservation of Heat* $$H = \int_{V(t)} \rho C_p T \, dV$$



$$\frac{d}{dt} \int_{V(t)} \rho C_p T \, dV = - \int_{V(t)} \nabla \cdot \mathbf{q} \, dV + \int_{V(t)} R \, dV$$

$$(\mathbf{q} = -k \nabla T)$$

* assumes constant ρ and C_p

Final Conservation Equation

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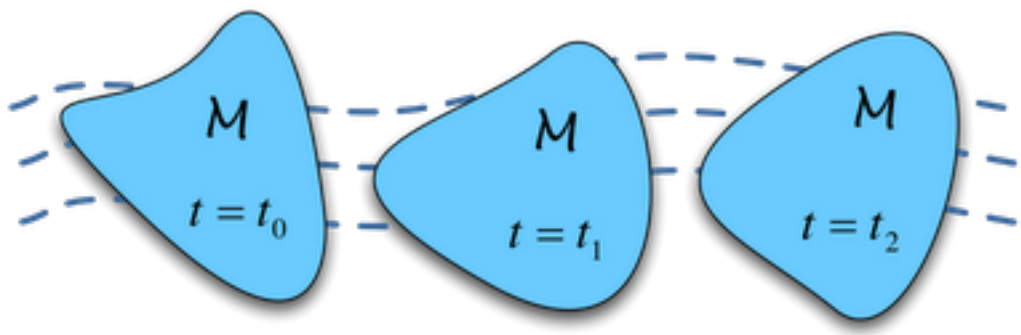

$$\int_{V(t)} \left(\rho C_p \frac{DT}{Dt} + \rho C_p T (\nabla \cdot \mathbf{v}) \right) dV = \int_{V(t)} \left(\nabla \cdot k \nabla T + R \right) dV$$

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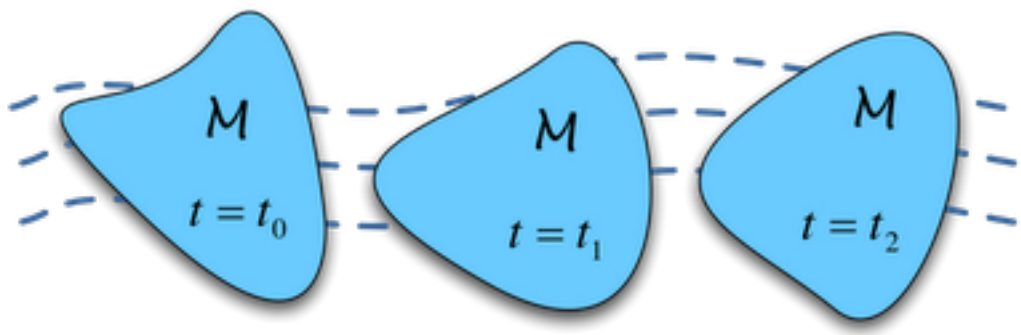
Final Conservation Equation

Conservation of Heat* $$H = \int_{V(t)} \rho C_p T \, dV$$



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* assumes constant ρ and C_p

Summary for Incompressible Fluid

mass $$\frac{D\rho}{Dt} = 0 \longrightarrow \nabla \cdot \mathbf{v} = 0$$

momentum $$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \eta \nabla^2 \mathbf{v} + \rho \mathbf{g}$$

heat $$\rho C_p \frac{DT}{Dt} = k \nabla^2 T + R$$

Subject to boundary conditions on **v** and T. Two useful parameters

$$\nu = \frac{\eta}{\rho} \approx 10^{18}\ \mathrm{m^2 s^{-1}} \qquad \kappa = \frac{k}{\rho C_P} \approx 10^{-6}\ \mathrm{m^2 s^{-1}}$$

Boussinesq Approximation

mass $$\frac{D\rho}{Dt} = 0 \longrightarrow \nabla \cdot \mathbf{v} = 0$$

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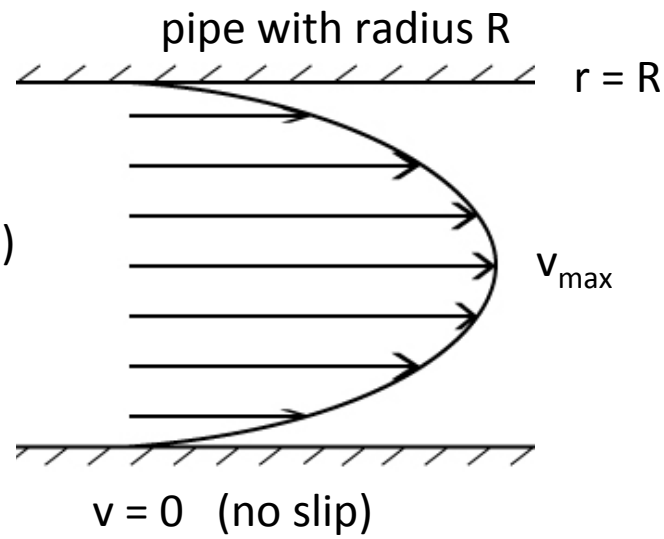
Equation of state: relate density variations to changes in temperature

$$\frac{1}{\rho}\frac{\partial \rho}{\partial T} = -\alpha$$ thermal expansion

Dimensionless Numbers

Why?!

We often describe quantities in SI units (m, kg, s, etc.)



Other choices

- distance L in terms of R
- velocity V in terms of v_{max}
- time t in terms of R/v_{max}

Order of Magnitudes:

$$\nabla \mathbf{v} \sim \frac{\mathbf{v}_{max} - 0}{R} \sim \mathcal{O}(1)$$

$$\nabla^2 \mathbf{v} \sim \frac{\mathbf{v}_{max}/R - 0}{R} \sim \frac{\mathbf{v}_{max}}{R^2} \sim \mathcal{O}(1)$$

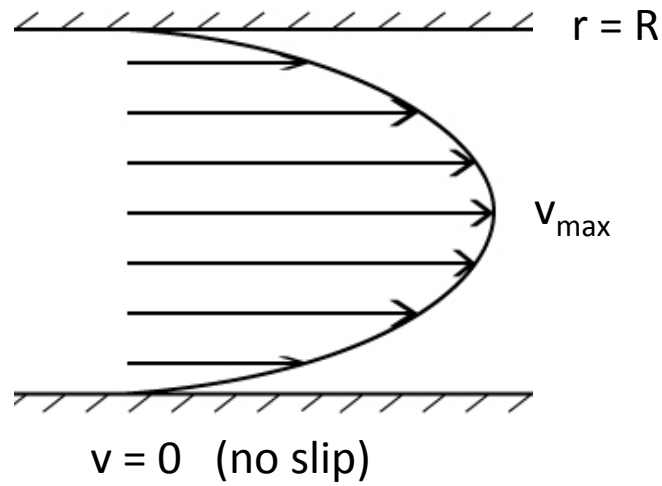
Dimensionless Numbers

Define dimensionless variables

$x' = x / R$

$v' = v / v_{max}$

$t' = t / \tau$



Navier-Stokes (in SI units)

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla P + \eta\nabla^2\mathbf{v}$$

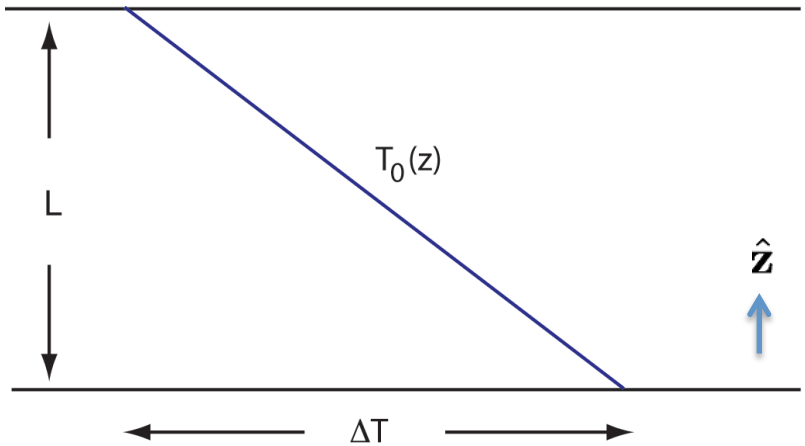
Change variables to x', t', v'

$$\left(\frac{\partial \mathbf{v}'}{\partial t'} + \mathbf{v}'\cdot\nabla\mathbf{v}'\right) = -\nabla P' + \frac{1}{Re}\nabla^2\mathbf{v}'$$

$$Re = \frac{v_{max}R}{(\eta/\rho)} = \frac{v_{max}R}{\nu}$$

Reynolds number

Convection Problem



Dimensionless variables

$x' = x / L$

$T' = T / \Delta T$

$t' = t / \tau$ where $\tau = L^2/\kappa$

$v' = v / (L/\tau)$

Change variables to dimensionless quantities

1. $\nabla \cdot \mathbf{v}' = 0$

2. $\frac{1}{Pr}\left(\frac{\partial \mathbf{v}'}{\partial t'} + \mathbf{v}' \cdot \nabla \mathbf{v}'\right) = -\nabla P' + RaT'\hat{\mathbf{z}} + \nabla^2 \mathbf{v}'$

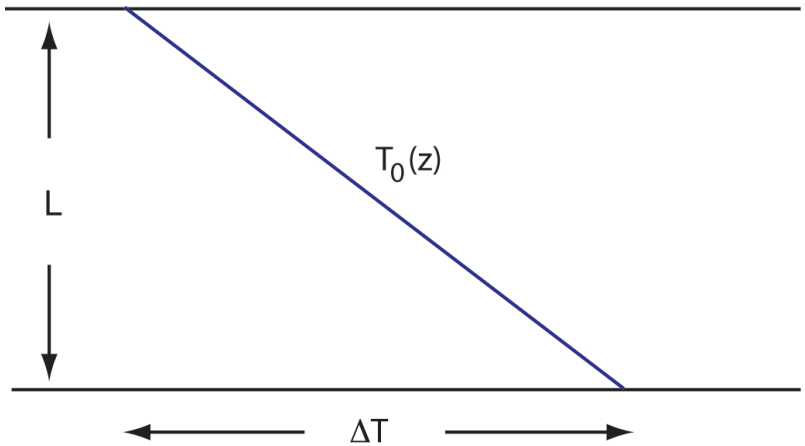
3. $\frac{\partial T'}{\partial t'} + \mathbf{v}' \cdot \nabla T' = \nabla^2 T' + R'$

Dimensionless numbers

$$Pr = \frac{\nu}{\kappa}$$

$$Ra = \frac{\alpha g \Delta T L^3}{\kappa \nu}$$

Convection Problem



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3. $\frac{\partial T'}{\partial t'} + \mathbf{v}' \cdot \nabla T' = \nabla^2 T' + R'$

Dimensionless numbers

$$Pr = \frac{\nu}{\kappa} \sim 10^{24}$$

(mantle)

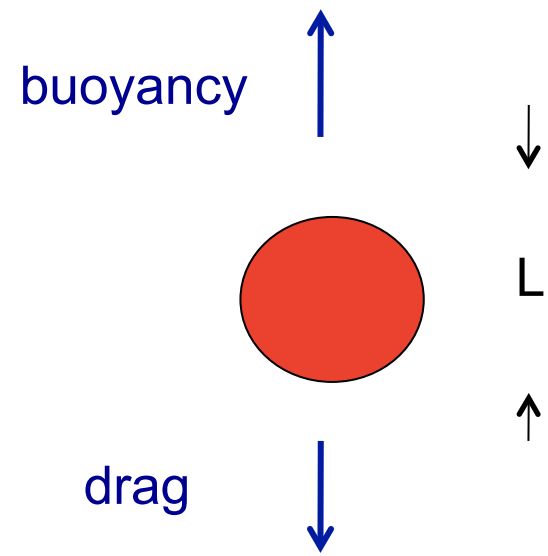
$$Ra = \frac{\alpha g \Delta T L^3}{\kappa \nu}$$

Interpretation of Rayleigh Number

Velocity of Parcel $v \approx \Delta\rho g L^2/\eta$

For hot fluid $|\Delta\rho| = \rho\alpha\Delta T$

Advection time $\tau_a = \dfrac{L}{v}$



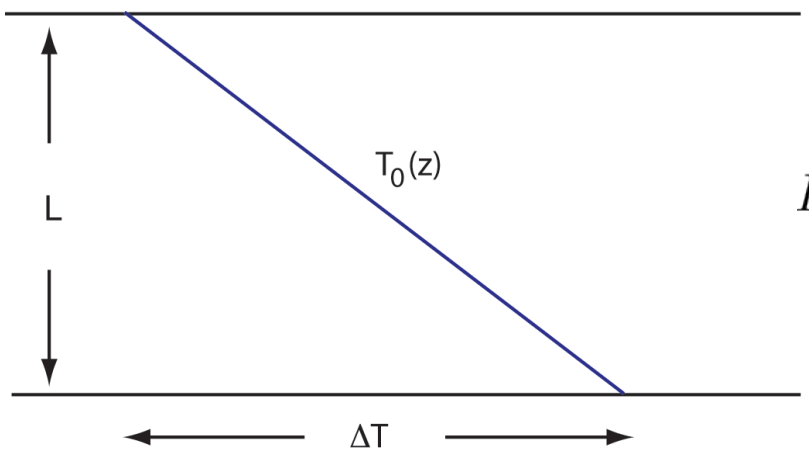
How does this compare to conduction time ($\tau_c = L^2/\kappa$) ?

$$\frac{\tau_c}{\tau_a} = \frac{vL}{\kappa} = \frac{\rho\alpha g \Delta T L^3}{\kappa\eta}$$ (Rayleigh number)

For Earth L = 2900 km, ΔT ~ 3000 K, Ra ~ 10^8 (critical $Ra_c = 10^3$)

Onset of Convection

When does convection begin?



$$Ra = \frac{\alpha g \Delta T L^3}{\kappa \nu}$$

Consider time evolution of a small perturbation in an initially conductive state

$$T(x, y, z, t) = T_0(z) + \delta T(x, y, z)\, e^{\sigma t}$$

$$\mathbf{v}(x, y, z, t) = \delta \mathbf{v}(x, y, z) e^{\sigma t}$$

Substitute into (linearized) equations and solve for growth rate σ

Onset of Convection

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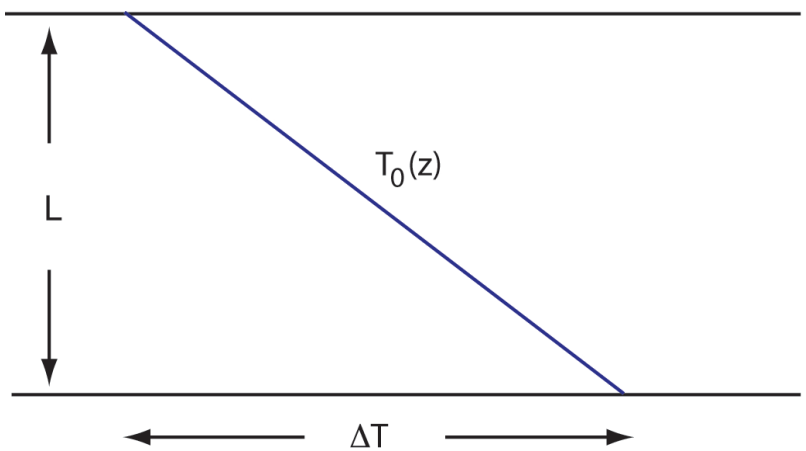

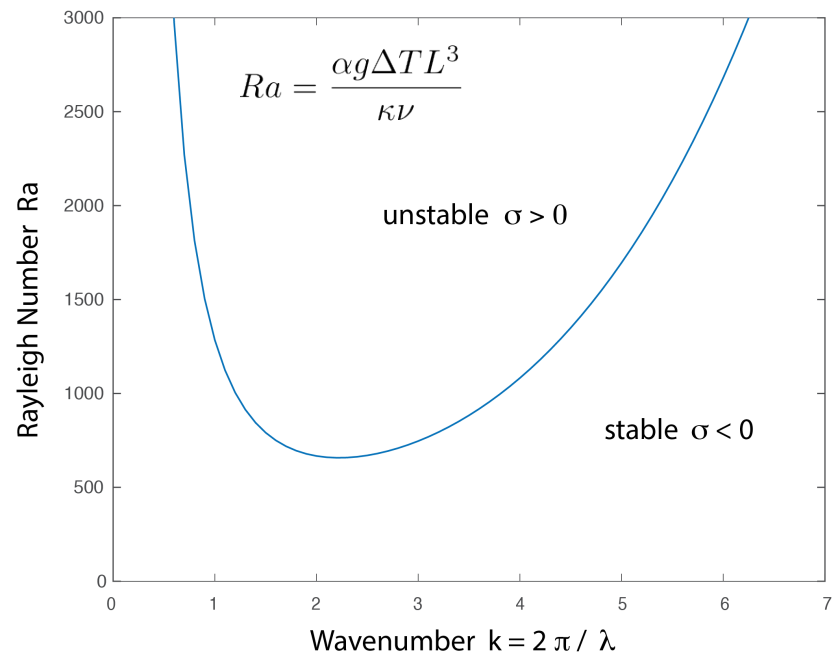

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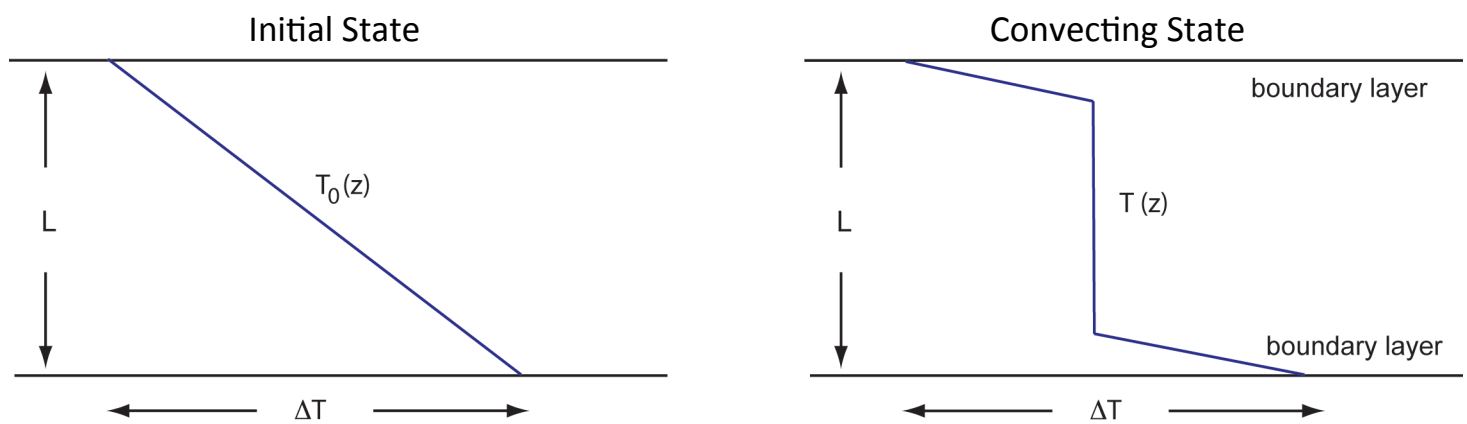
Substitute into (linearized) equations and solve for growth rate σ

Boundary Layer Theory



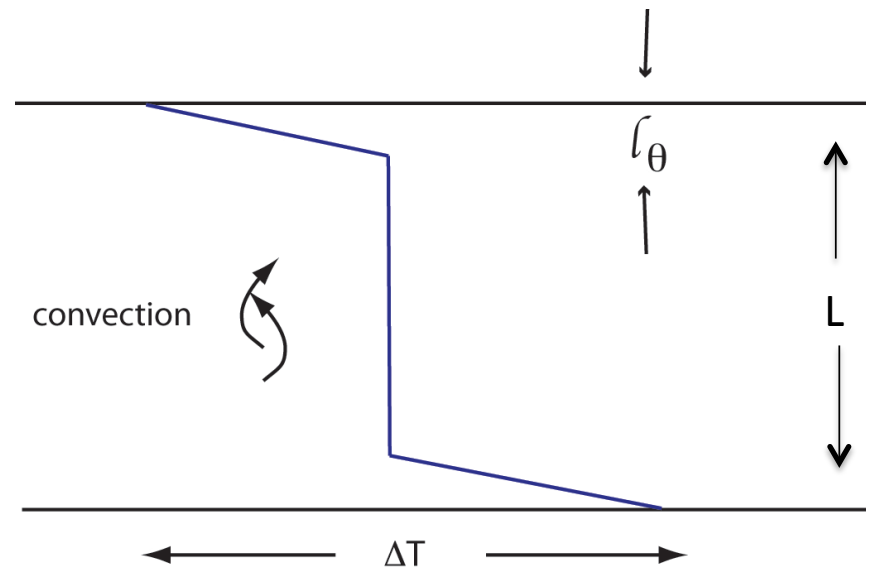
Heat is carried by advection in the interior (e.g. $q_z = \rho\ Cp\ T\ v_z$). The vertical velocity vanishes at the boundaries, so heat must be carried by conduction across the boundaries (e.g. $q_z = -\ k\ dT/dz$).

→ The boundary layers are key to understanding convection

Boundary Layer Theory

Heat flow across layer*

$$q_{conv} = \frac{k(\Delta T/2)}{l_\theta}$$



In the initial state (before convection)

$$q_{cond} = \frac{k(\Delta T)}{L}$$

Efficiency of convection $\frac{q_{conv}}{q_{cond}} = \frac{L}{2l_\theta} = Nu$ (Nusselt number)

* l_θ is average value

Boundary Layer Instabilities

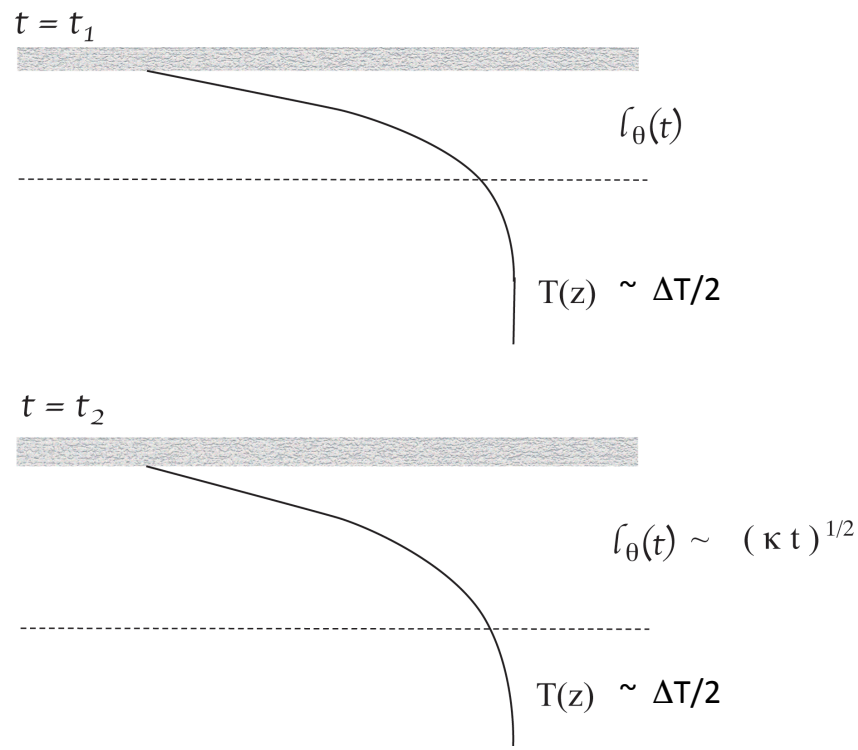
Cold boundary layer grows by conduction into the convecting region

$$l_\theta \approx \sqrt{\kappa t}$$

Eventually the boundary layer becomes unstable at time t_c

Define a local Rayleigh number

$$Ra_l = \frac{\alpha g (\Delta T/2)\, l_\theta^3}{\kappa \nu}$$

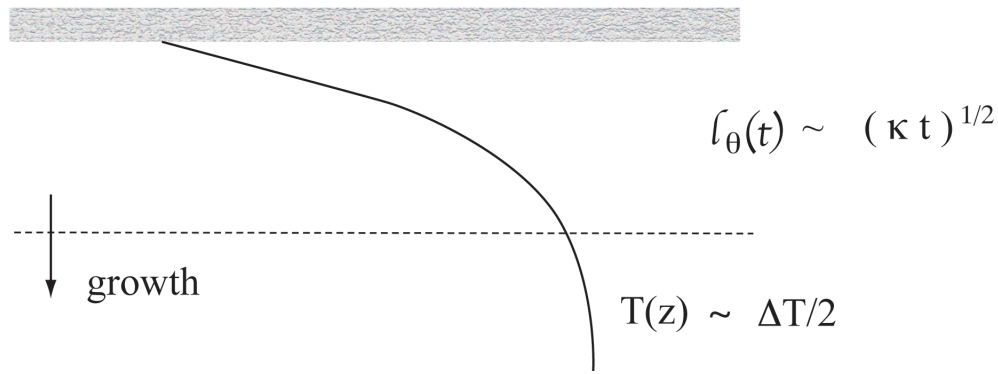


Instability occurs when $Ra_l \sim Ra_c \sim 10^3$

Average Heat Flow

Heat flow q(t)

$$q(t) = -k\frac{dT}{dz} \approx k\frac{\Delta T/2}{\sqrt{\kappa t}}$$



Time average

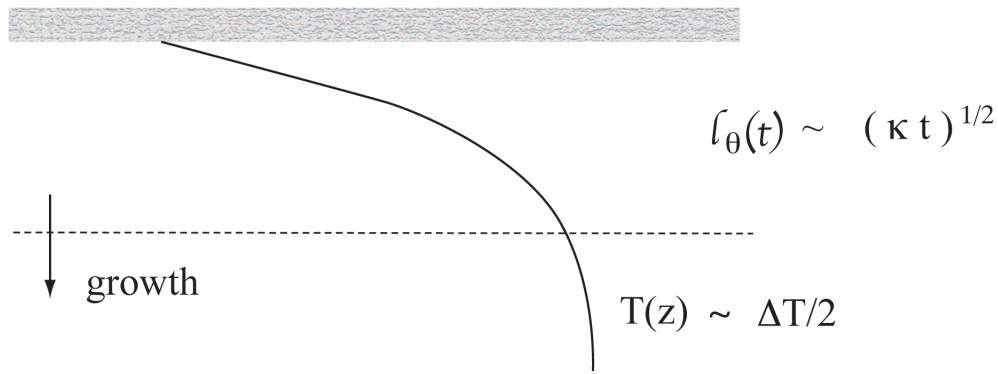
$$\bar{q} \approx \frac{1}{t_c}\int_0^{t_c} k\frac{\Delta T/2}{\sqrt{\kappa t}}\,dt = \frac{k\Delta T}{\sqrt{\kappa t_c}}$$

Recall that $l_\theta^c = \sqrt{\kappa t_c}$ is defined by $Ra_l = Ra_c$

Nu-Ra Relationship

Time average

$$\bar{q} = k\Delta T / l_\theta^c$$



where

$$Ra_c = \frac{\alpha(\Delta T/2) g \, (l_\theta^c)^3}{\kappa\nu} = \frac{Ra}{2}\left(\frac{l_\theta^c}{L}\right)^3$$

This means that

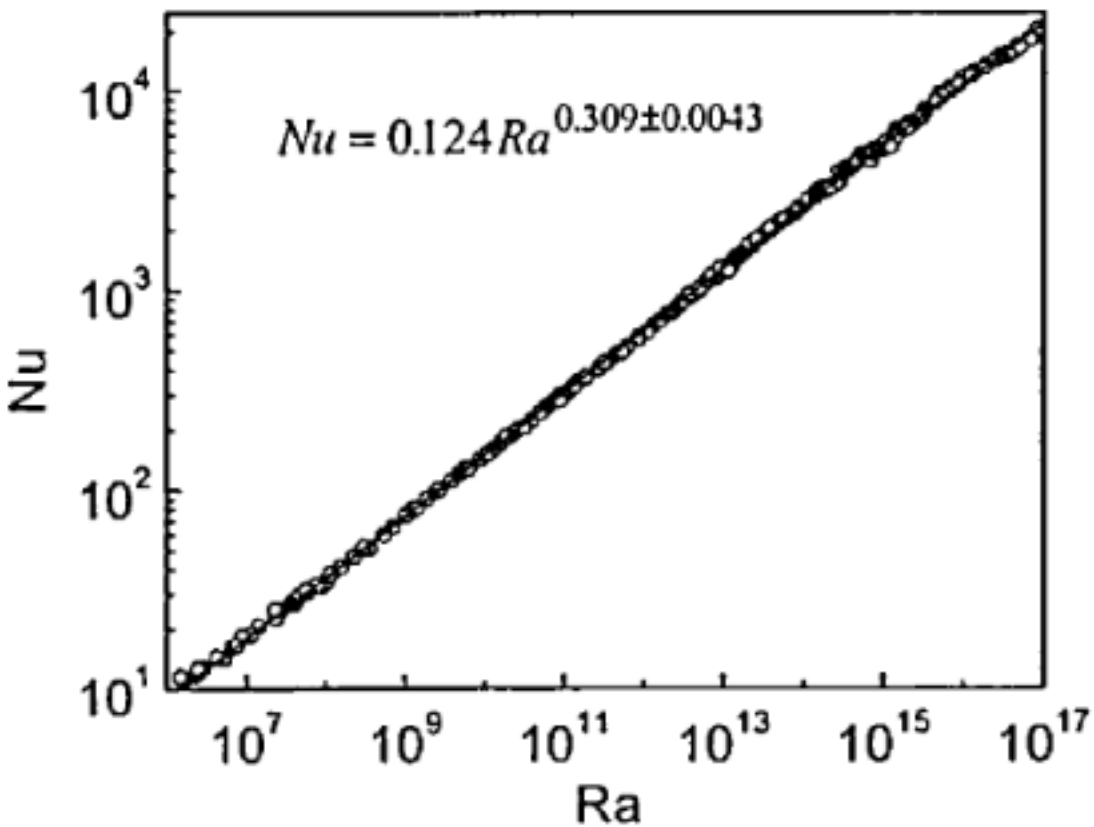
$$\frac{l_\theta^c}{L} = \left(\frac{2Ra_c}{Ra}\right)^{1/3} \quad \Rightarrow \quad Nu = \frac{L}{2\bar{l}_\theta} = \left(\frac{Ra}{2Ra_c}\right)^{1/3}$$

* remember that $l_\theta^c = 2\bar{l}_\theta$

Nu-Ra relationship

Comparison with Experiments

Experiments (Niemela et al. 2000)



Existence of asymptotic regime? Nu ~ $Ra^{1/2}$

Boundary-layer theory

$$Nu = \left(\frac{Ra}{2Ra_c} \right)^{1/3}$$

$$Nu = 0.089 Ra^{0.33}$$

using Kraichnan's estimate for Ra_c ~ 700

Application to Mantle Convection

1. Thickness of lithospheric plates

$$\frac{l_\theta^c}{L} = \left(\frac{2Ra_c}{Ra}\right)^{1/3}$$

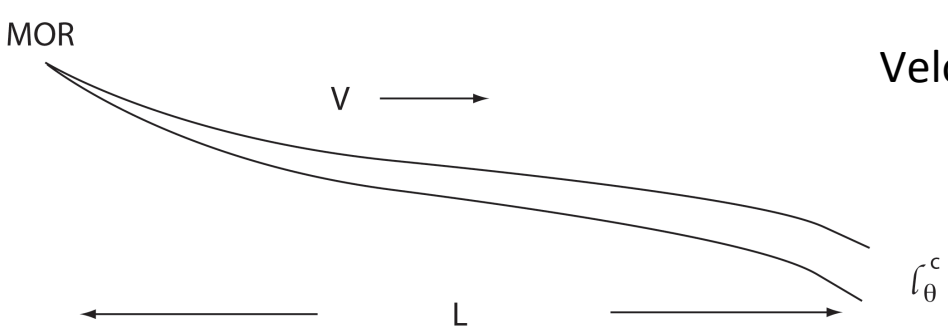
for Ra = 10^8, Ra_c = 10^3, L = 2900 km we get l_θ = 80 km

2. Velocity of lithosphere

Cooling time $$t_c = \frac{l_\theta^2}{\kappa}$$

Velocity $$V = \frac{L}{t_c} = \left(\frac{Ra}{2Ra_c}\right)^{2/3} \frac{\kappa}{L}$$

~ 1.5 cm/year



Application to Mantle Convection

3. Heat Flow

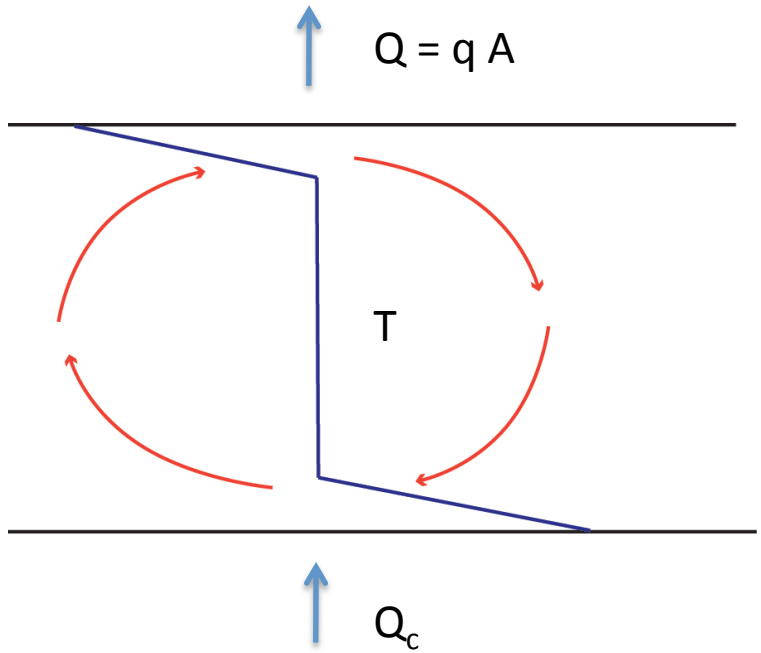
$$q = \left(\frac{k\Delta T}{L}\right) Nu \quad \text{where} \quad Nu = \left(\frac{Ra}{2Ra_c}\right)^{1/3}$$

for Ra = 10^8, Ra_c = 10^3, L = 2900 km we get Nu ~ 36

Taking k = 3 W m^{-1} K^{-1}, ΔT = 3000 K yields q_{cond} = 3 mW m^{-2} or Q_{cond} = 1.4 TW

→ Q_{conv} ~ 50 TW (instead of 46 TW)

Thermal Histories



Heat Budget

$$\bar{C}_p M \frac{dT}{dt} = R(t) - Q(t) + Q_c(t)$$

Parameterized Convection

$$q(t) = \frac{kT(t)}{L} Nu(t) = \frac{kT(t)}{L} \left(\frac{Ra(t)}{2Ra_c} \right)^{1/3}$$

where

$$Ra(t) = \frac{\alpha g T(t) L^3}{\kappa \nu(t)}$$

Temperature Dependence

$$\nu(T) \propto \exp\left(\frac{E}{RT} \right)$$

Further Complications

Extensions to boundary layer theory?

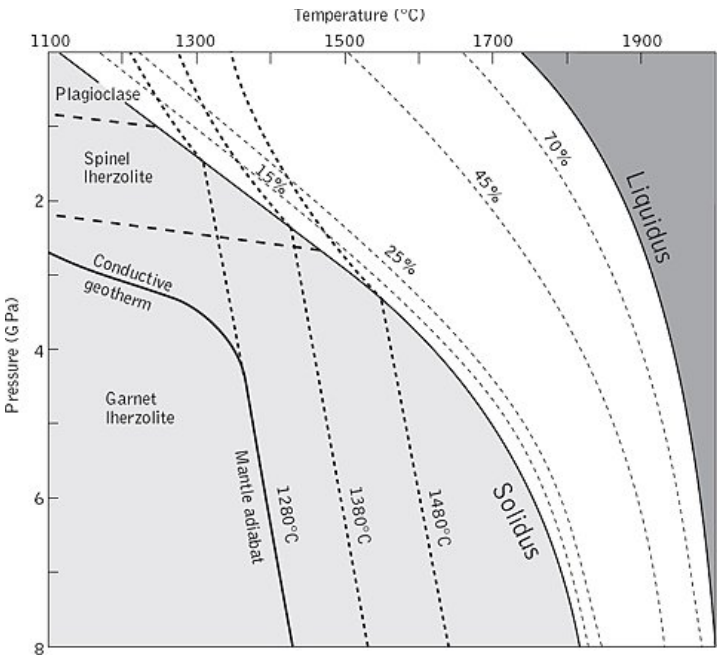
1. Melting and volcanism - affects heat transport and composition (buoyancy)

2. Rheology – some mixture of elastic, viscous and plastic behavior

3. Initial condition?

Melting

Decompression Melting



Melting forms oceanic crust (basalt) and depleted residuum (harzburgite)

Densities

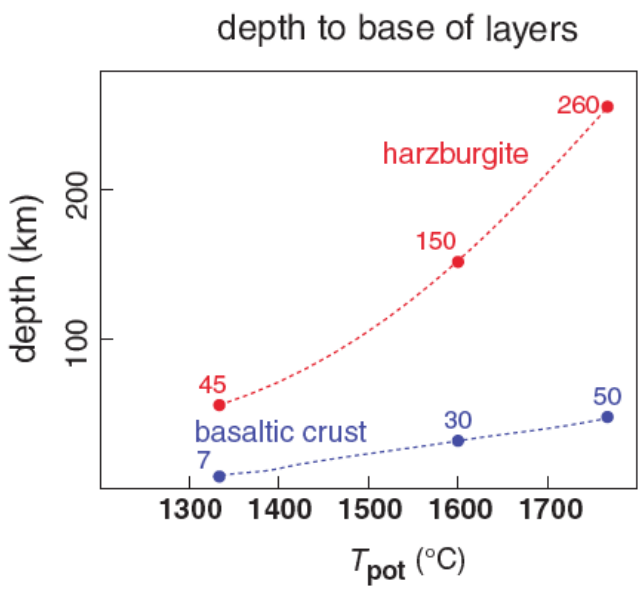
basalt ~ 2.9 g/cm^3

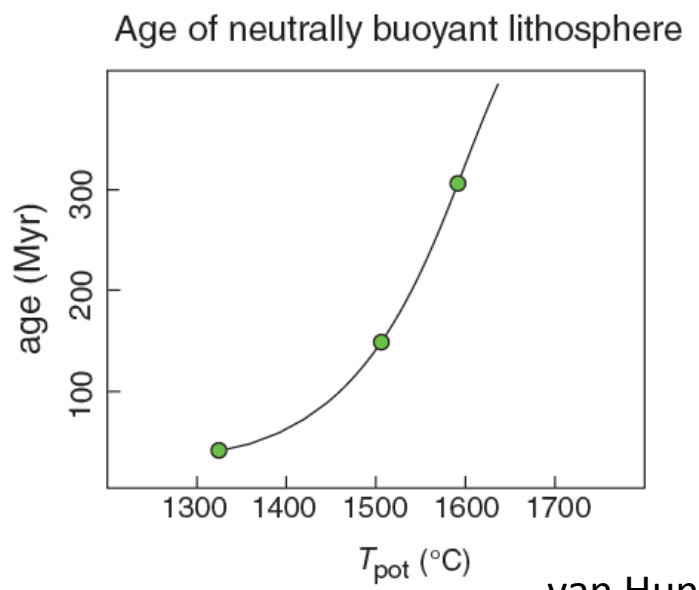
harzburgite ~ 3.2 g/cm^3

lherzolite ~ 3.3 g/cm^3

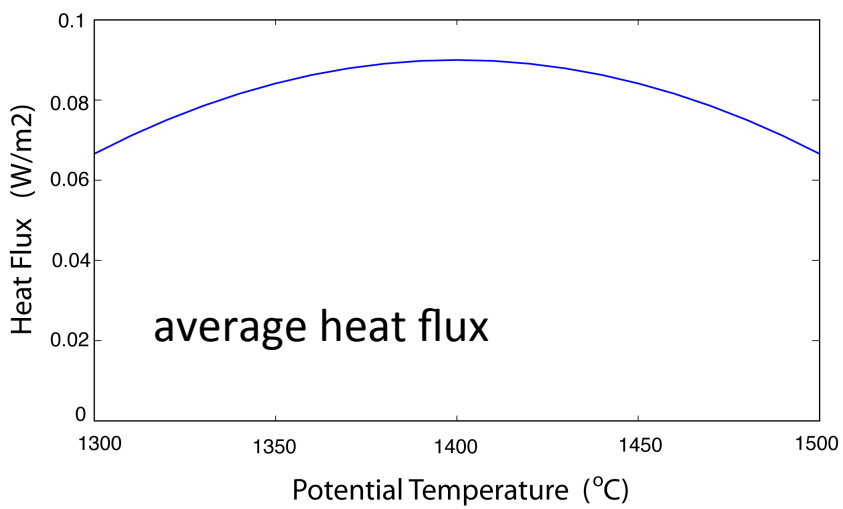
Oxburgh & Parmentier (1977)

Buoyancy of Lithosphere





van Hunen et al. (2008)



Sleep (2007)

Rheology

Surface Force $\mathbf{T} = -pI + \tau$ where τ is deviatoric stress tensor

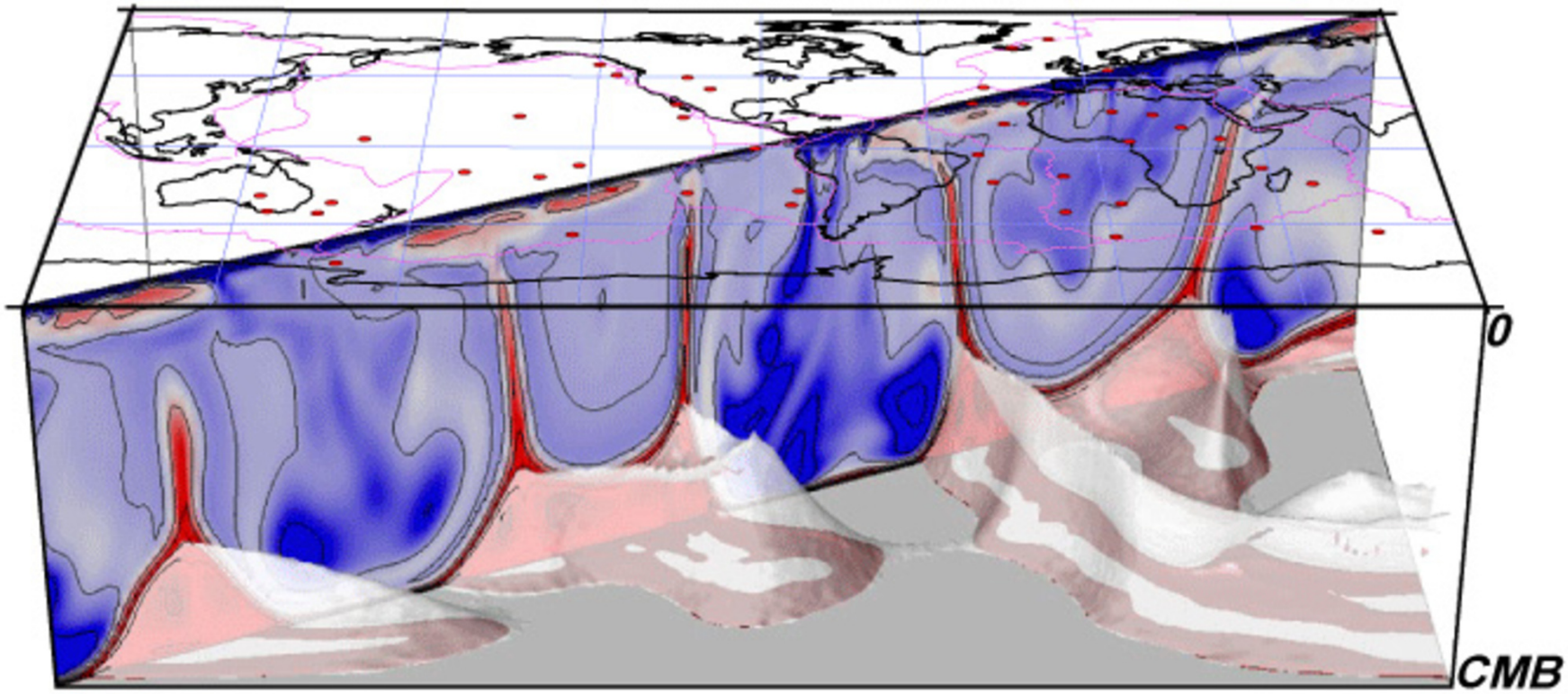
Newtonian fluid $\tau = 2\eta\dot{\epsilon}$ where strain rate $\dot{\epsilon} = \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)$

More general flow laws

$$\dot{\epsilon} = f(T, P, \tau, ...)\tau$$

Other variables: grain size, volatiles, partial melt, deformation history

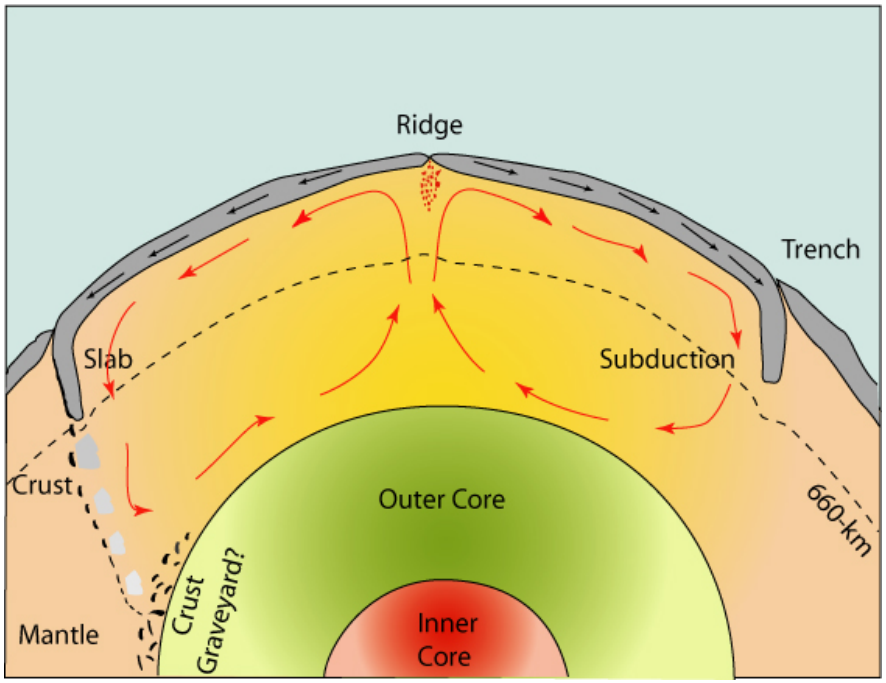
Primordial Layering?



McNamara et al. (2010)

Summary

How good is our theory of mantle convection?



(S. Rost)

Extrapolation back in time ?

- heat flow?
- number or size of plates?
- continental configuration?
- surface environment/climate?