Geodynamics III
Mantle Convection

Quelle & Schmelling (2002)
Convection Everywhere
Convection Everywhere

Solar Convection  Simulation

Abbett et al. 2004

Mixing length model
Outline

1. A brief overview of the governing equations
2. Introduction to the Rayleigh number $Ra$
3. Onset of convection (elements of stability)
4. The boundary layer model
5. Scaling relations and thermal histories
Conservation Equations

Fluid Parcel (mass $M$ or volume $V$)

Reynolds Transport Theorem

\[
\frac{d}{dt} \int_{V(t)} f \, dV = \int_{V(t)} \left[ \frac{\partial f}{\partial t} + \nabla \cdot (v f) \right] \, dV
\]
Conservation Equations

Example: Conservation of Mass

\[ M = \int_{V(t)} \rho \, dV \]

\[ \frac{d}{dt} \int_{V(t)} \rho \, dV = \int_{V(t)} \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v} \rho) \right] \, dV = 0 \]
Conservation Equations

Conservation of mass requires

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v} \rho) = 0$$

Other (equivalent) forms

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho (\nabla \cdot \mathbf{v}) = 0$$

$$\frac{D\rho}{Dt} + \rho (\nabla \cdot \mathbf{v}) = 0$$

material derivative
Conservation Equations

Example: Conservation of Momentum

\[ p = \int_{V(t)} \rho v \, dV \]

Newton’s 2nd Law

\[ \frac{d}{dt} \int_{V(t)} \rho v \, dV = F \]

total force on parcel V

(e.g. gravity, pressure, viscous drag, etc)
Conservation Equations

Example: Conservation of Heat*

\[ H = \int_{V(t)} \rho C_p T \, dV \]

\[ \frac{d}{dt} \int_{V(t)} \rho C_p T \, dV = -\int_{S(t)} \mathbf{q} \cdot d\mathbf{S} + \int_{V(t)} R \, dV \]

* assumes constant \( \rho \) and \( C_p \)

conduction across surface \( S(t) \) \( (\mathbf{q} = -k \nabla T) \)
Summary for Incompressible Fluid

mass
\[ \frac{D\rho}{Dt} = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{v} = 0 \]

momentum
\[ \rho \frac{D\mathbf{v}}{Dt} = -\nabla P + \rho \mathbf{g} + \eta \nabla^2 \mathbf{v} \]

heat
\[ \rho C_p \frac{DT}{Dt} = k \nabla^2 T + R \]

Boussinesq approximation for small density variations

\[ \Delta \rho = -\rho \alpha \Delta T \text{ included in buoyancy term only} \]
Modes of Heat Transport

1. Conduction
   
   time scale \( \tau_c = \frac{L^2}{\kappa} \)
   
   where \( \kappa = \frac{k}{\rho \ C_p} \) is thermal diffusivity

2. Advection
   
   time scale \( \tau_a = \frac{L}{v} \)
   
   Relative importance
   
   \[ \frac{\tau_c}{\tau_a} = \frac{\nu L}{\kappa} \]

   e.g. \( L = 2900 \ \text{km}, \ \nu \sim 10\ \text{cm/year}, \ \kappa = 10^{-6} \ \text{m}^2/\text{s} \)
   \[ \tau_c/\tau_a \sim 1000 \]
Rayleigh Number  $Ra$

Velocity of Parcel  

$v \approx \Delta \rho g L^2 / \eta$

For hot fluid  

$|\Delta \rho| = \rho \alpha \Delta T$

Ratio of conduction to advection time?

$$\frac{\tau_c}{\tau_a} = \frac{vL}{\kappa} = \frac{\rho \alpha g \Delta T L^3}{\kappa \eta}$$  

(Rayleigh number)

e.g.  $L = 2900$ km, $\Delta T \sim 3000$ K, $Ra \sim 10^8$  (critical $Ra_c \sim 10^3$)
Onset of Convection

When does convection begin?

Consider time evolution of a small perturbation in an initially conductive state

\[ T(x, y, z, t) = T_0(z) + \delta T(x, y, z) e^{\sigma t} \]

\[ \mathbf{v}(x, y, z, t) = \delta \mathbf{v}(x, y, z) e^{\sigma t} \]

Substitute into (linearized) equations and solve for growth rate \( \sigma \)
Critical Rayleigh Number

Calculate growth rate

- use solutions at three times \((t_1, t_2, t_3)\)

\[
\frac{T(t_3) - T(t_1)}{T(t_2) - T(t_1)} = e^{\sigma(t_3 - t_2)}
\]

Rearrange for \(\sigma\)

\[
\sigma = \frac{1}{t_3 - t_2} \ln \left( \frac{T(t_3) - T(t_1)}{T(t_2) - T(t_1)} \right)
\]

Plot result as a function of \(Ra\)
Heat is carried by advection in the interior (e.g. $q_z = \rho \, C_p \, T \, v_z$). The vertical velocity vanishes at the boundaries, so heat must be carried by conduction across the boundaries (e.g. $q_z = -k \, dT/dz$).

→ The boundary layers are key to understanding convection
Boundary Layer Theory

Heat flow across layer

\[ q_{\text{conv}} = \frac{k(\Delta T/2)}{l_\theta} \]

In the initial state (before convection)

\[ q_{\text{cond}} = \frac{k(\Delta T)}{L} \]

Efficiency of convection

\[ \frac{q_{\text{conv}}}{q_{\text{cond}}} = \frac{L}{2l_\theta} = Nu \]  

(Nusselt number)

\* \( l_\theta \) is average value
Boundary Layer Instabilities

Cold boundary layer grows by conduction into the convecting region

\[ l_\theta \approx \sqrt{\kappa t} \]

Eventually the boundary layer becomes unstable at time \( t_c \)

Define a local Rayleigh number

\[ Ra_l = \frac{\alpha g (\Delta T/2) l_\theta^3}{\kappa \nu} \]

Instability occurs when \( Ra_l \sim Ra_c \sim 10^3 \)
Average Heat Flow

Heat flow \( q(t) \)

\[
q(t) = -k \frac{dT}{dz} \approx k \frac{\Delta T/2}{\sqrt{\kappa t}}
\]

Time average

\[
\bar{q} \approx \frac{1}{t_c} \int_0^{t_c} k \frac{\Delta T/2}{\sqrt{\kappa t}} dt = \frac{k \Delta T}{\sqrt{\kappa t_c}}
\]

Recall that \( l_\theta^c = \sqrt{\kappa t_c} \) is defined by \( Ra_l = Ra_c \)
Nu-Ra Relationship

Time average

\[ \bar{q} = \frac{k \Delta T}{l^c_\theta} \]

where

\[ Ra_c = \frac{\alpha(\Delta T/2)g (l^c_\theta)^3}{\kappa \nu} = \frac{Ra}{2} \left( \frac{l^c_\theta}{L} \right)^3 \]

This means that

\[ \frac{l^c_\theta}{L} = \left( \frac{2Ra_c}{Ra} \right)^{1/3} \]

\[ Nu = \left( \frac{Ra}{2Ra_c} \right)^{1/3} \]

* remember that \( l^c_\theta = 2\bar{l}_\theta \)

Nu-Ra relationship
Application to Mantle Convection

1. Thickness of lithospheric plates

\[ \frac{l_\theta^c}{L} = \left( \frac{2Ra_c}{Ra} \right)^{1/3} \]

for \( Ra = 10^8 \), \( Ra_c = 10^3 \), \( L = 2900 \) km we get \( l_\theta = 80 \) km

2. Velocity of lithosphere

Cooling time

\[ t_c = \frac{l_\theta^2}{\kappa} \]

Velocity

\[ V = \frac{L}{t_c} = \left( \frac{Ra}{2Ra_c} \right)^{2/3} \frac{\kappa}{L} \]

\( \sim 1.5 \) cm/year
Use of Energy Equations

Momentum equation

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla P - \alpha T \mathbf{g} + \nu \nabla^2 \mathbf{v}
\]

Kinetic energy equation

\[
\int_V \mathbf{v} \cdot \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{1}{\rho} \nabla P + \ldots \right) dV = 0
\]

Time Average

\[
\varepsilon_v = \frac{\nu^3}{L^4} (Nu - 1) Ra Pr^{-2}
\]

where \( Pr = \frac{\nu}{\kappa} \) is the Prandtl number and \( \varepsilon_v \) is the viscous dissipation (e.g.

\[
\varepsilon_v \equiv \frac{1}{V} \int_V \nu (\nabla v)^2 dV
\]
Approximate viscous dissipation

\[ \epsilon_v \equiv \frac{1}{V} \int_v \nu (\nabla v)^2 \, dV \]

\[ \approx \nu \left( \frac{v}{L} \right)^2 \]

Use in Time Average

\[ \epsilon_v = \frac{\nu^3}{L^4} (Nu - 1) Ra Pr^{-2} \]

\[ v = \frac{\kappa}{L} (Nu - 1)^{1/2} Ra^{1/2} \]

\[ v \approx 0.25 Ra^{2/3} \frac{\kappa}{L} \]

using boundary layer theory \( Nu = (Ra/2Ra_c)^{1/3} \)
Turbulent Convection

Turbulent cascade to small scales

\[ \varepsilon_v \equiv \frac{1}{V} \int_V \nu (\nabla v)^2 \, dV \]

\[ \approx \frac{v^2}{\tau} \approx \frac{v^3}{L} \]

Use in Time Average

\[ \varepsilon_v = \frac{v^3}{L^4} (Nu - 1) Ra Pr^{-2} \]

\[ v \approx \frac{\kappa}{L} (Pr Nu Ra)^{1/3} \]
Mixing Length Model

Temperature Equation

\[
\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \kappa \nabla^2 T
\]

“Thermal” Power

\[
\int_V T \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T - \kappa \nabla^2 T \right) dV = 0
\]

Time Average

\[
\epsilon_\theta = \frac{\kappa}{L^2} \Delta T^2 \ Nu
\]

where \( \epsilon_\theta = \frac{1}{V} \int_V \kappa (\nabla T)^2 dV \)

Letting \( \epsilon_\theta = \nu \Delta T^2 / L \)

\[
Nu \approx Pr^{1/2} Ra^{1/2}
\]
Thermal Histories

Heat Budget

\[ \bar{C}_p M \frac{dT}{dt} = R(t) - Q(t) \]

Convection

\[ q(t) = \frac{kT(t)}{L} N_u(t) = \frac{kT(t)}{L} \left( \frac{Ra(t)}{2Ra_c} \right)^{1/3} \]

where

\[ Ra(t) = \frac{\alpha g T(t) L^3}{\kappa \nu(t)} \]

Temperature Dependence

\[ \nu(T) \propto \exp\left( \frac{E}{RT} \right) \]
Changes in Heat Flow

Viscosity

Heat Flow

Strong temperature dependence leads to a thermal “catastrophe” at early times

argument for high Urey ratio
How is Mantle Convection Different?

Decompression Melting

Melting forms oceanic crust (basalt) and depleted residuum (harzburgite)

Densities

- basalt $\sim 2.9 \text{ g/cm}^3$
- harzburgite $\sim 3.2 \text{ g/cm}^3$
- lherzolite $\sim 3.3 \text{ g/cm}^3$

Oxburgh & Parmentier (1977)
Buoyancy of Lithosphere

van Hunen et al. (2008)

Sleep (2007)
Rheology of Lithosphere

Melting dehydrates and strengthens the lithosphere (Korenaga, 2010)

Viscous Dissipation includes

- internal viscosity

- lithosphere “viscosity”

\[ \nu(t) = \left( \frac{Ra}{2Ra_c} \right)^{1/3} \Delta \eta_L(h)^{-1/3} \]

A problem for the magnetic field?
Rheology of Lithosphere

Bending Stress

Bending Moment

Power law

\[ \dot{\epsilon}_{ij} = \frac{1}{2\eta} \left( \sigma_{II}^{n-1} \right) \sigma'_{ij} \]

(Buffett & Becker, 2012)
Summary

We can make sense of mantle convection using boundary layer theory

Extrapolation back in time?

- heat flow?
- number or size of plates?
- continental configuration?
- surface environment/climate?

(S. Rost)