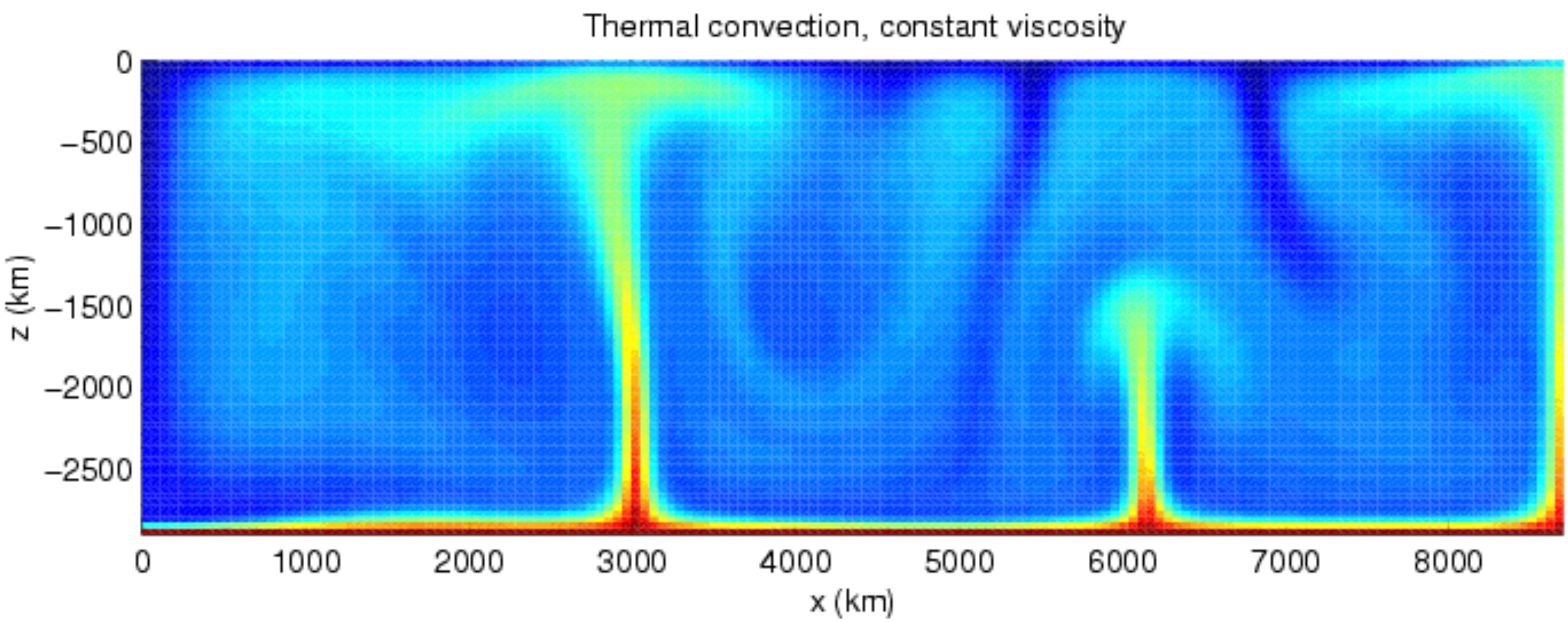
Geodynamics III
Mantle Convection



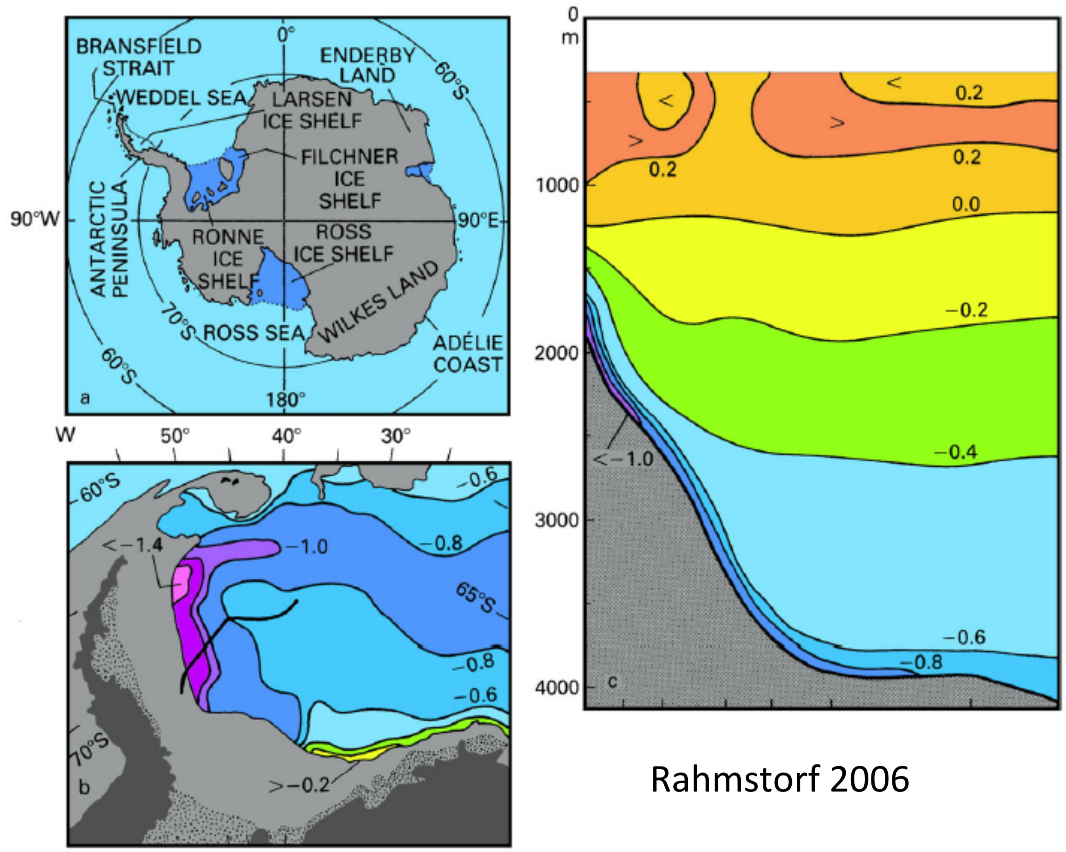
Quelle & Schmelling (2002)

Convection Everywhere

Atmosphere

NOAA

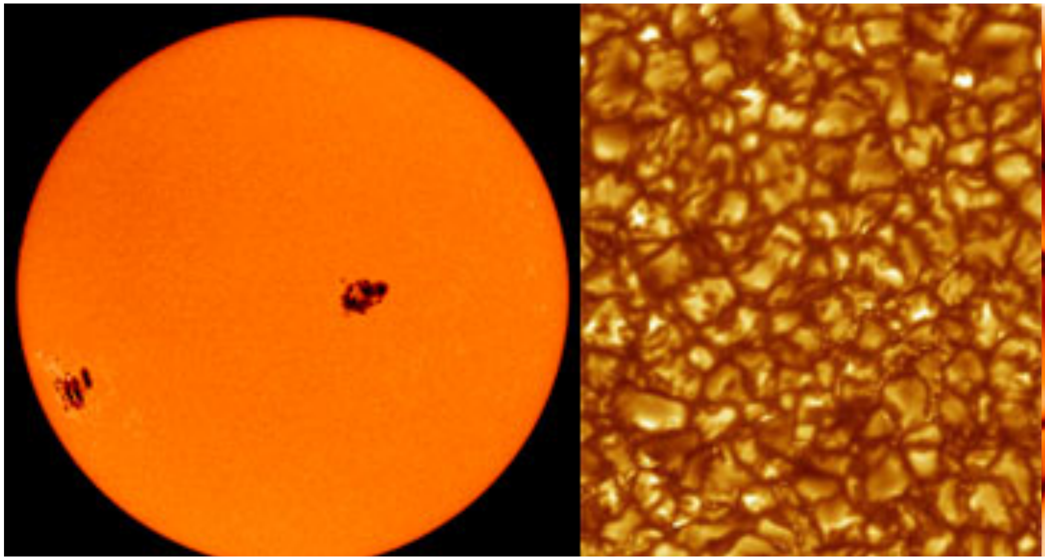
Ocean



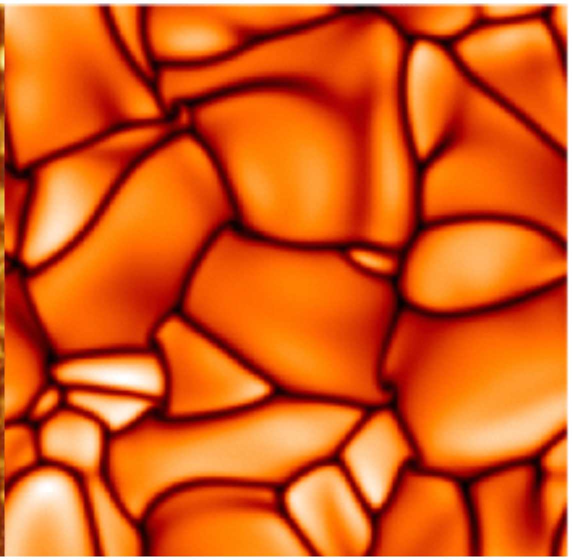
Rahmstorf 2006

Convection Everywhere

Solar Convection



Simulation



Abbett et al. 2004

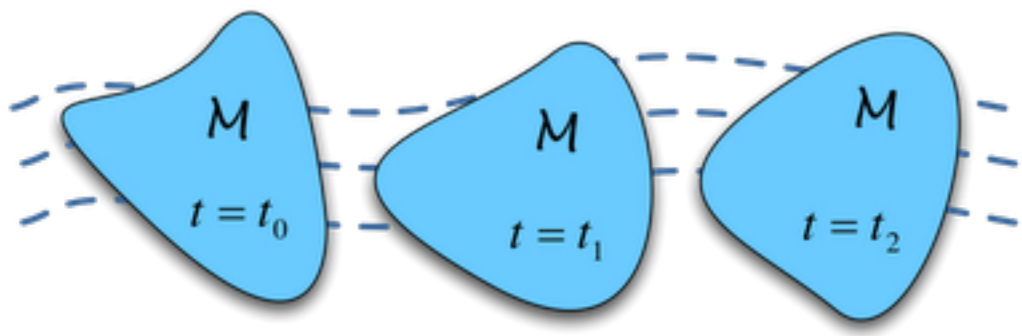
Mixing length model

Outline

1. A brief overview of the governing equations
2. Introduction to the Rayleigh number Ra
3. Onset of convection (elements of stability)
4. The boundary layer model
5. Scaling relations and thermal histories

Conservation Equations

Fluid Parcel (mass M or volume V)

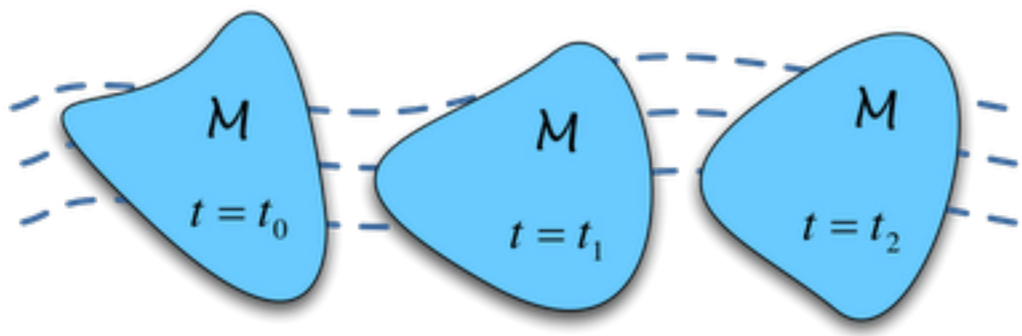


Reynolds Transport Theorem

$$\frac{d}{dt}\int_{V(t)} f\,dV = \int_{V(t)} \left[\frac{\partial f}{\partial t} + \nabla\cdot(\mathbf{v}f)\right] dV$$

Conservation Equations

Example: Conservation of Mass $M = \int_{V(t)} \rho \, dV$



$$\frac{d}{dt}\int_{V(t)} \rho \, dV = \int_{V(t)} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v}\rho)\right] dV = 0$$

Conservation Equations

Conservation of mass requires

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v}\rho) = 0$$

Other (equivalent) forms

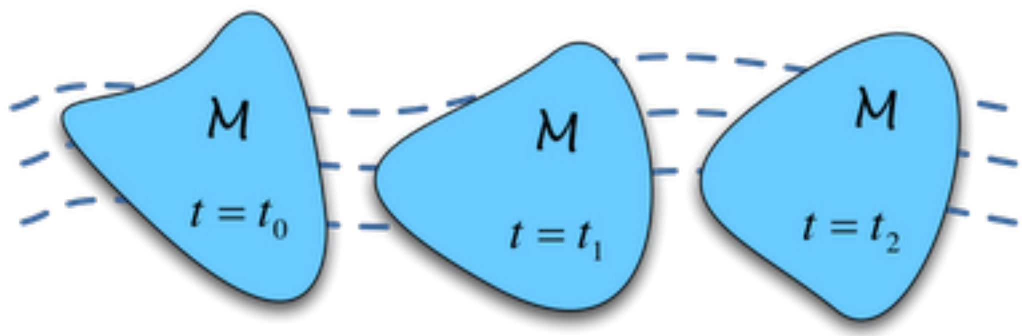
$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho(\nabla \cdot \mathbf{v}) = 0$$

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0$$

material derivative

Conservation Equations

Example: Conservation of Momentum $\mathbf{p} = \int_{V(t)} \rho \mathbf{v} \, dV$



Newton's 2nd Law

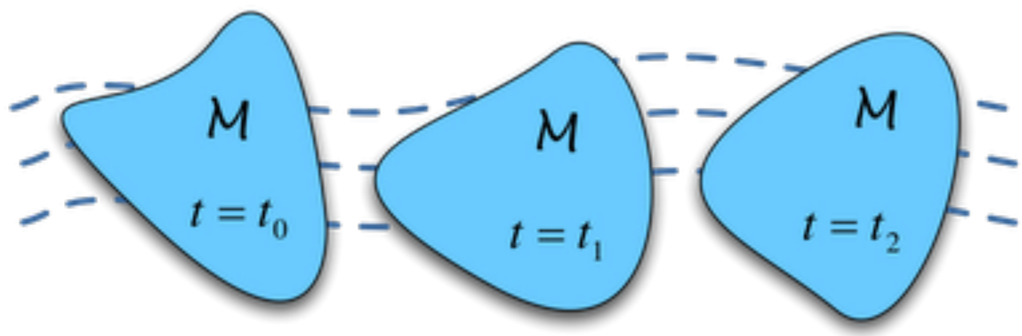
$$\frac{d}{dt} \int_{V(t)} \rho \mathbf{v} \, dV = \mathbf{F}$$

total force on parcel V

(e.g. gravity, pressure, viscous drag, etc)

Conservation Equations

Example: Conservation of Heat* $H = \int_{V(t)} \rho C_p T \, dV$



$$\frac{d}{dt} \int_{V(t)} \rho C_p T \, dV = - \int_{S(t)} \mathbf{q} \cdot \mathbf{dS} + \int_{V(t)} R \, dV$$

conduction across surface S(t) $(\mathbf{q} = -k \nabla T)$

* assumes constant ρ and C_p

Summary for Incompressible Fluid

mass $$\frac{D\rho}{Dt} = 0 \quad \longrightarrow \quad \nabla \cdot \mathbf{v} = 0$$

momentum $$\rho \frac{D\mathbf{v}}{Dt} = -\nabla P + \rho \mathbf{g} + \eta \nabla^2 \mathbf{v}$$

heat $$\rho C_p \frac{DT}{Dt} = k \nabla^2 T + R$$

Boussinesq approximation for small density variations

→ variations $\Delta\rho = -\rho \, \alpha \, \Delta T$ included in buoyancy term only

Modes of Heat Transport

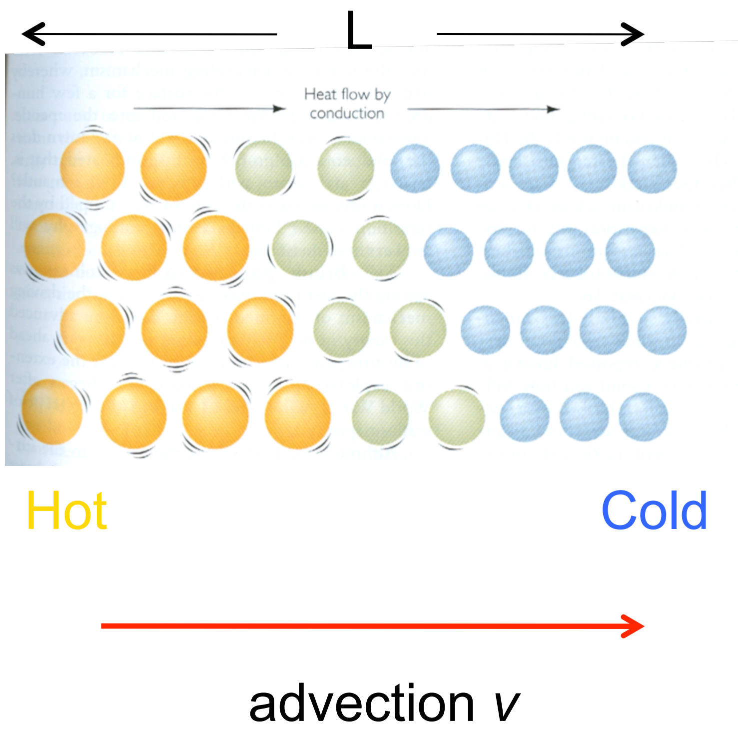
1. Conduction

time scale $$\tau_c = \frac{L^2}{\kappa}$$

where κ = k/ ρ Cp is thermal diffusivity

2. Advection

time scale $$\tau_a = \frac{L}{v}$$

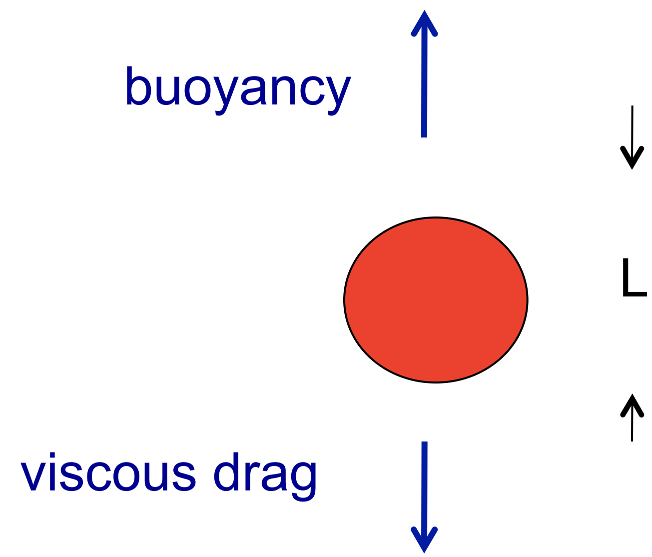


Relative importance $$\frac{\tau_c}{\tau_a} = \frac{vL}{\kappa}$$

e.g. L = 2900 km, v ~ 10cm/year, $\kappa = 10^{-6}\ m^2/s$ → τ_c/τ_a ~ 1000

Rayleigh Number Ra

Velocity of Parcel $v \approx \Delta\rho g L^2/\eta$



For hot fluid $|\Delta\rho| = \rho\alpha\Delta T$

Ratio of conduction to advection time?

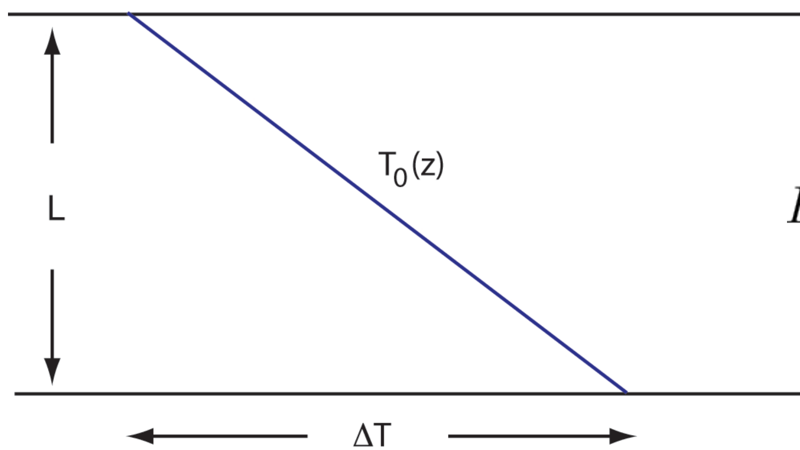
$$\frac{\tau_c}{\tau_a} = \frac{vL}{\kappa} = \frac{\rho\alpha g\Delta T L^3}{\kappa\eta}$$

(Rayleigh number)

e.g. L = 2900 km, $\Delta T \sim$ 3000 K, Ra ~ 10^8 (critical $Ra_c \sim 10^3$)

Onset of Convection

When does convection begin?



$$Ra = \frac{\alpha g \Delta T L^3}{\kappa \nu}$$

Consider time evolution of a small perturbation in an initially conductive state

$$T(x, y, z, t) = T_0(z) + \delta T(x, y, z)\, e^{\sigma t}$$

$$\mathbf{v}(x, y, z, t) = \delta \mathbf{v}(x, y, z) e^{\sigma t}$$

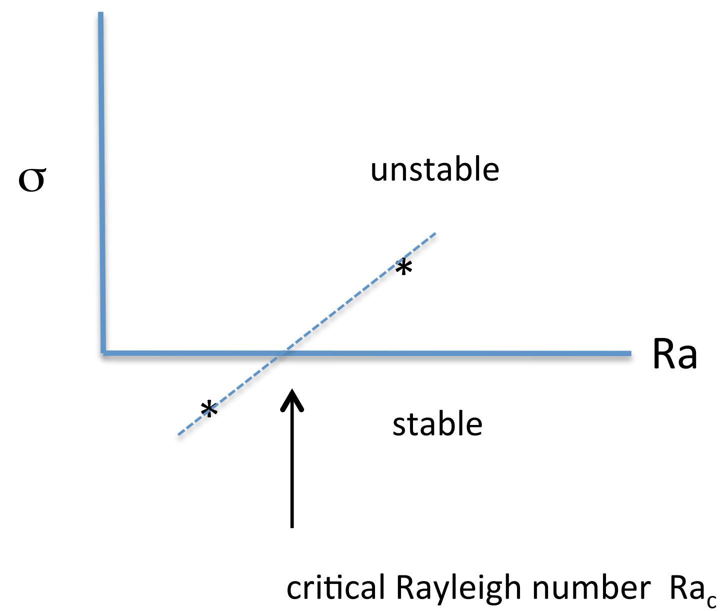
Substitute into (linearized) equations and solve for growth rate σ

Critical Rayleigh Number

Calculate growth rate

- use solutions at three times (t_1, t_2, t_3)

$$\frac{T(t_3) - T(t_1)}{T(t_2) - T(t_1)} = e^{\sigma(t_3 - t_2)}$$

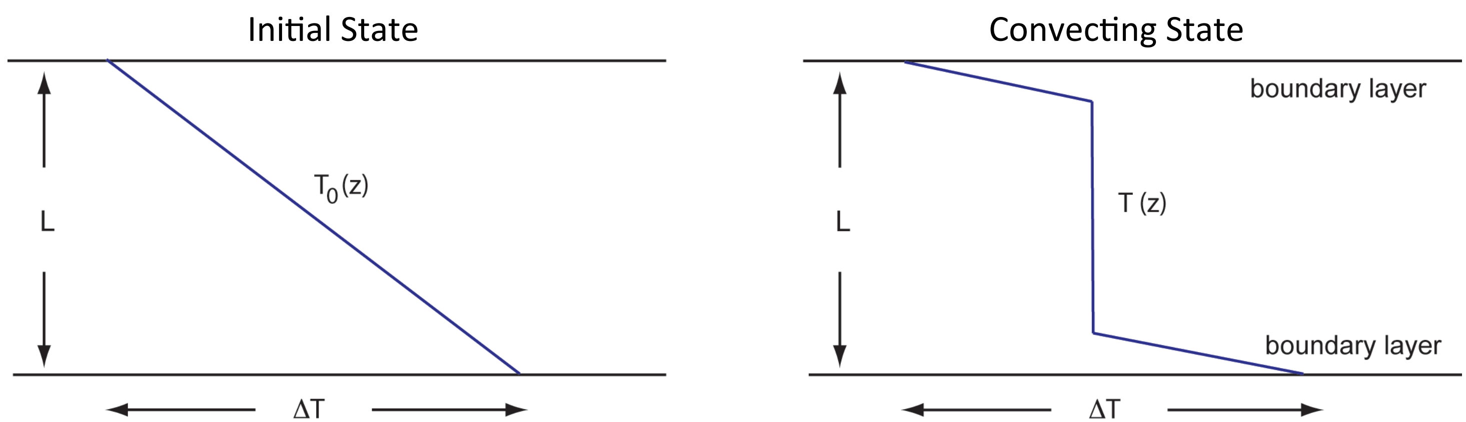


Rearrange for σ

$$\sigma = \frac{1}{t_3 - t_2} \ln\left(\frac{T(t_3) - T(t_1)}{T(t_2) - T(t_1)}\right)$$

Plot result as a function of Ra

Boundary Layer Theory



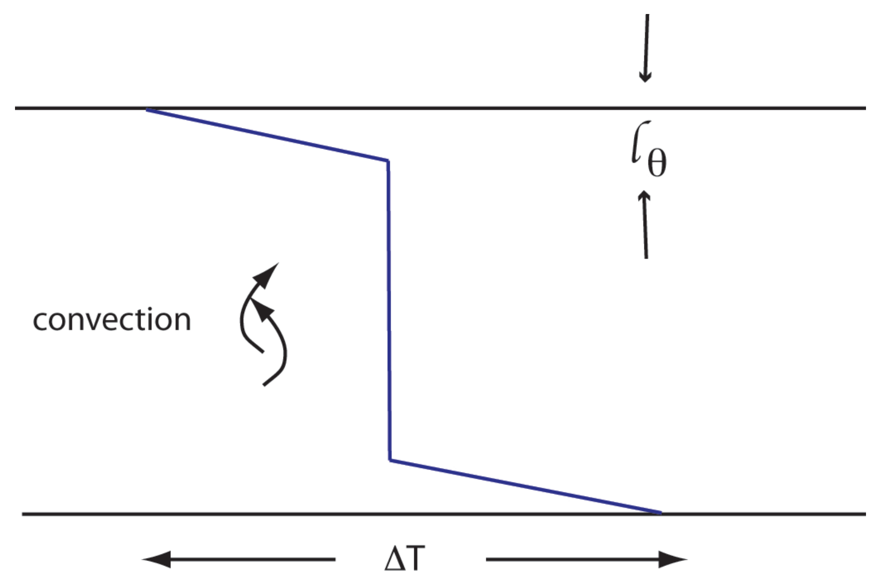
Heat is carried by advection in the interior (e.g. $q_z = \rho\ Cp\ T\ v_z$). The vertical velocity vanishes at the boundaries, so heat must be carried by conduction across the boundaries (e.g. $q_z = -\ k\ dT/dz$).

→ The boundary layers are key to understanding convection

Boundary Layer Theory

Heat flow across layer

$$q_{conv} = \frac{k(\Delta T/2)}{l_\theta}$$



In the initial state (before convection)

$$q_{cond} = \frac{k(\Delta T)}{L}$$

Efficiency of convection $$\frac{q_{conv}}{q_{cond}} = \frac{L}{2l_\theta} = Nu$$ (Nusselt number)

* l_θ is average value

Boundary Layer Instabilities

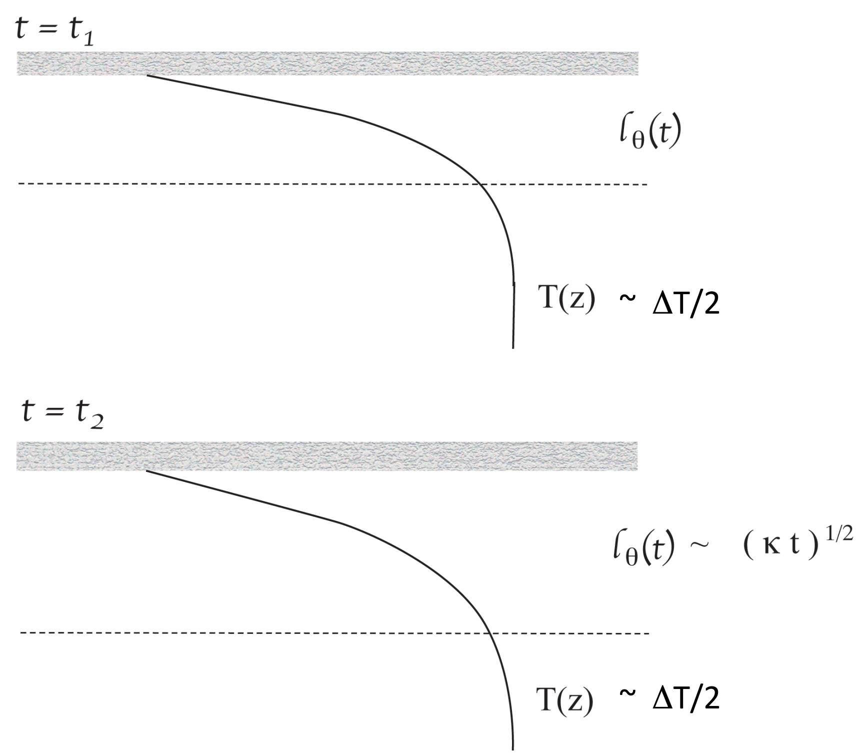
Cold boundary layer grows by conduction into the convecting region

$$l_\theta \approx \sqrt{\kappa t}$$

Eventually the boundary layer becomes unstable at time t_c

Define a local Rayleigh number

$$Ra_l = \frac{\alpha g(\Delta T/2)\, l_\theta^3}{\kappa \nu}$$

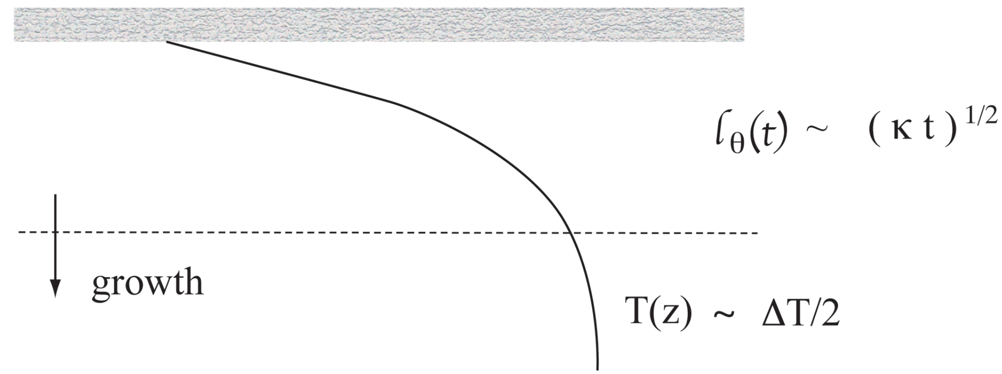


Instability occurs when $Ra_l \sim Ra_c \sim 10^3$

Average Heat Flow

Heat flow q(t)

$$q(t) = -k\frac{dT}{dz} \approx k\frac{\Delta T/2}{\sqrt{\kappa t}}$$



Time average

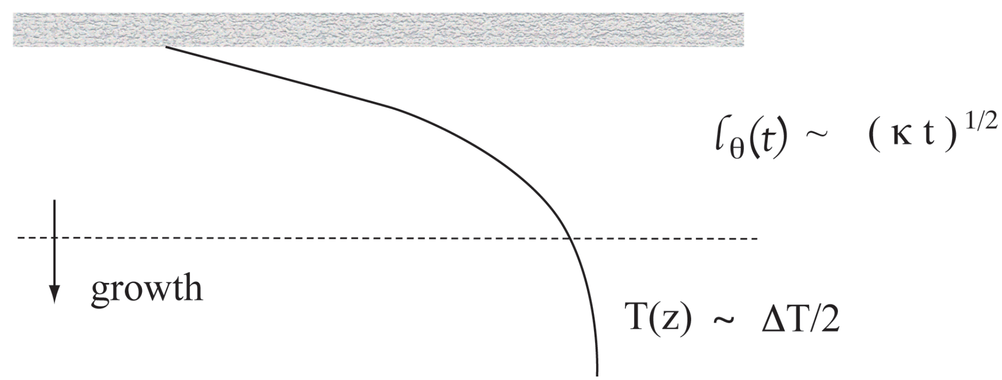
$$\bar{q} \approx \frac{1}{t_c}\int_0^{t_c} k\frac{\Delta T/2}{\sqrt{\kappa t}}\,dt = \frac{k\Delta T}{\sqrt{\kappa t_c}}$$

Recall that $l_\theta^c = \sqrt{\kappa t_c}$ is defined by $Ra_l = Ra_c$

Nu-Ra Relationship

Time average

$$\bar{q} = k\Delta T / l_\theta^c$$



where

$$Ra_c = \frac{\alpha(\Delta T/2)g\,(l_\theta^c)^3}{\kappa\nu} = \frac{Ra}{2}\left(\frac{l_\theta^c}{L}\right)^3$$

This means that

$$\frac{l_\theta^c}{L} = \left(\frac{2Ra_c}{Ra}\right)^{1/3} \quad \Rightarrow \quad Nu = \left(\frac{Ra}{2Ra_c}\right)^{1/3}$$

* remember that $l_\theta^c = 2\bar{l}_\theta$

Nu-Ra relationship

Application to Mantle Convection

1. Thickness of lithospheric plates

$$\frac{l_\theta^c}{L} = \left(\frac{2Ra_c}{Ra}\right)^{1/3}$$

for Ra = 10^8, $Ra_c = 10^3$, L = 2900 km we get l_θ = 80 km

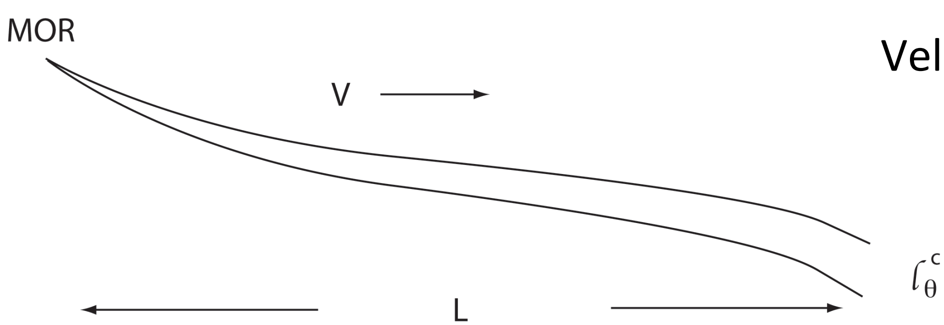
2. Velocity of lithosphere

Cooling time $$t_c = \frac{l_\theta^2}{\kappa}$$

Velocity $$V = \frac{L}{t_c}$$

$$= \left(\frac{Ra}{2Ra_c}\right)^{2/3} \frac{\kappa}{L}$$

~ 1.5 cm/year



Use of Energy Equations

Momentum equation

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla P - \alpha T \mathbf{g} + \nu \nabla^2 \mathbf{v}$$

Kinetic energy equation

$$\int_V \mathbf{v} \cdot \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{1}{\rho} \nabla P + ... \right) dV = 0$$

Time Average

$$\epsilon_v = \frac{\nu^3}{L^4} (Nu - 1) Ra Pr^{-2}$$

where $Pr = \nu/\kappa$ is the Prandtl number and ε_v is the viscous dissipation (e.g.

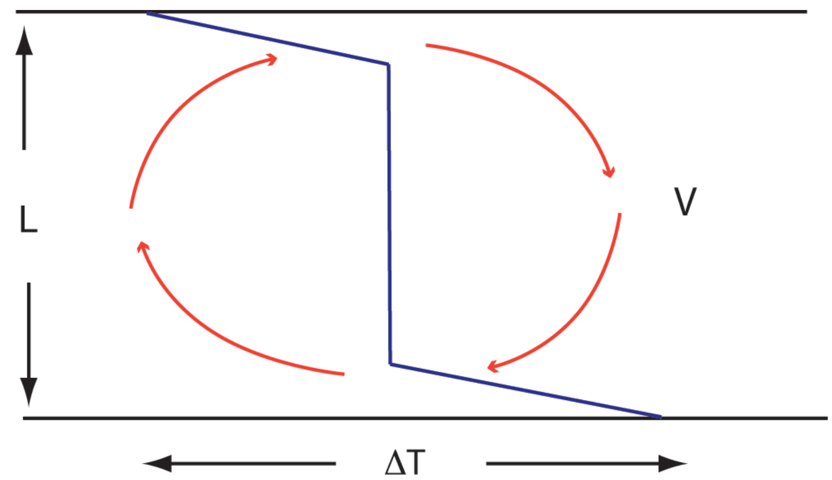
$$\epsilon_v \equiv \frac{1}{V} \int_v \nu (\nabla v)^2 \, dV$$

Illustration

Approximate viscous dissipation

$$\epsilon_v \equiv \frac{1}{V}\int_v \nu(\nabla v)^2\, dV$$

$$\approx \nu\left(\frac{v}{L}\right)^2$$



Use in Time Average

$$\epsilon_v = \frac{\nu^3}{L^4}(Nu-1)RaPr^{-2} \quad \longrightarrow \quad v = \frac{\kappa}{L}(Nu-1)^{1/2}Ra^{1/2}$$

using boundary layer theory $Nu = (Ra/2Ra_c)^{1/3}$

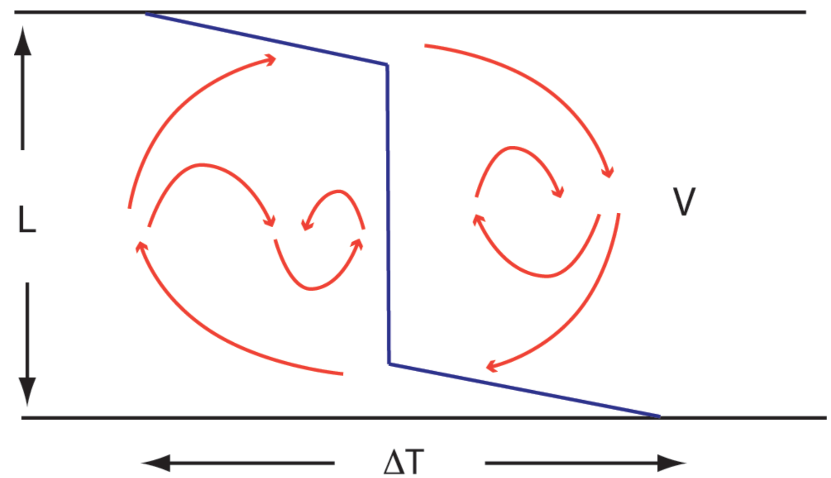
$$v \approx 0.25 Ra^{2/3}\frac{\kappa}{L}$$

Turbulent Convection

Turbulent cascade to small scales

$$\epsilon_v \equiv \frac{1}{V} \int_v \nu (\nabla v)^2 \, dV$$

$$\approx \frac{v^2}{\tau} \approx \frac{v^3}{L}$$



Use in Time Average

$$\epsilon_v = \frac{\nu^3}{L^4}(Nu - 1) Ra Pr^{-2} \quad \longrightarrow \quad v \approx \frac{\kappa}{L}(Pr\, Nu\, Ra)^{1/3}$$

Mixing Length Model

Temperature Equation

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \kappa \nabla^2 T$$

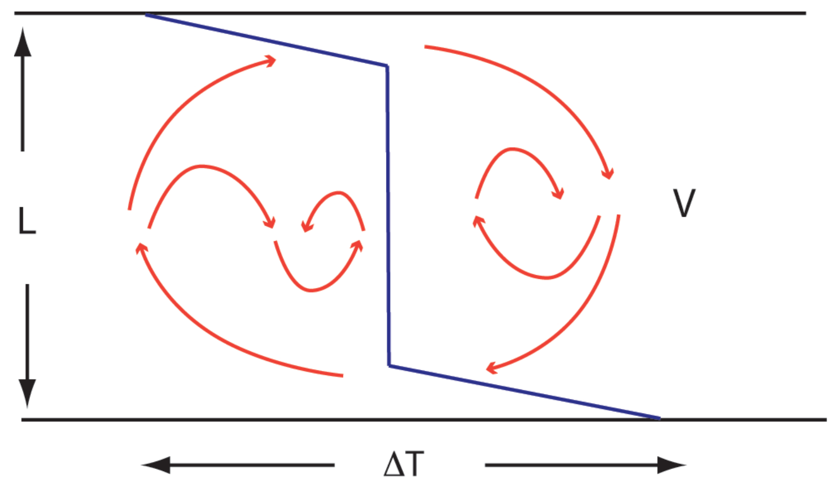
"Thermal" Power

$$\int_V T \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T - \kappa \nabla^2 T \right) dV = 0$$

Time Average

$$\epsilon_\theta = \frac{\kappa}{L^2} \Delta T^2 Nu$$

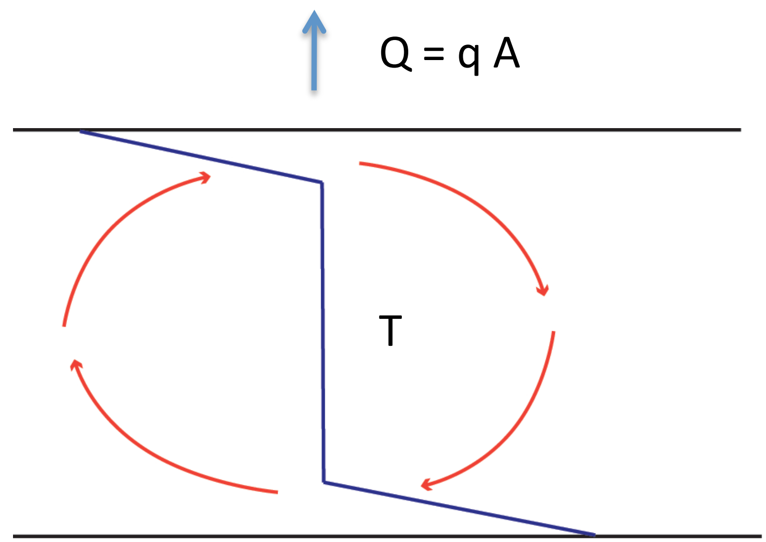
where $\epsilon_\theta = \frac{1}{V} \int_V \kappa (\nabla T)^2 \, dV$



Letting $\epsilon_\theta = v \, \Delta T^2 / L$

$$Nu \approx Pr^{1/2} Ra^{1/2}$$

Thermal Histories



Heat Budget

$$\bar{C}_p M \frac{dT}{dt} = R(t) - Q(t)$$

Convection

$$q(t) = \frac{kT(t)}{L} Nu(t) = \frac{kT(t)}{L} \left(\frac{Ra(t)}{2Ra_c} \right)^{1/3}$$

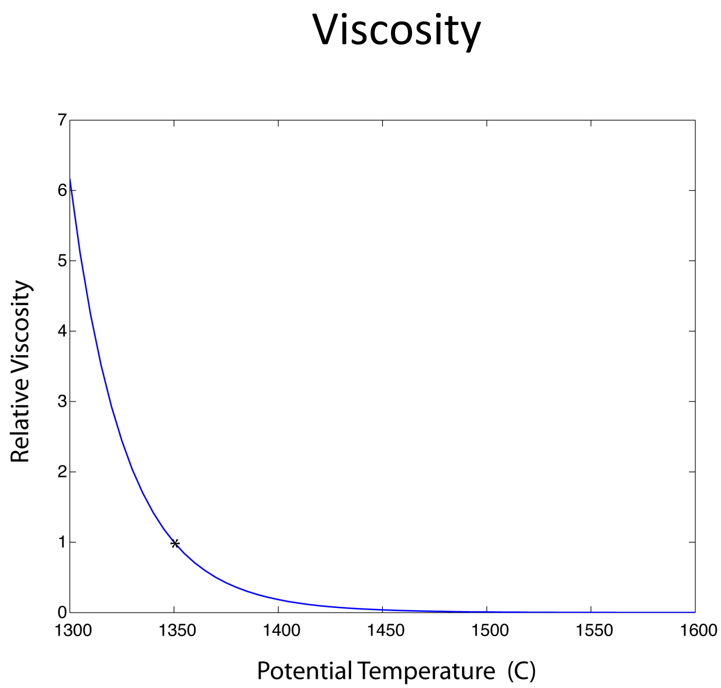
where

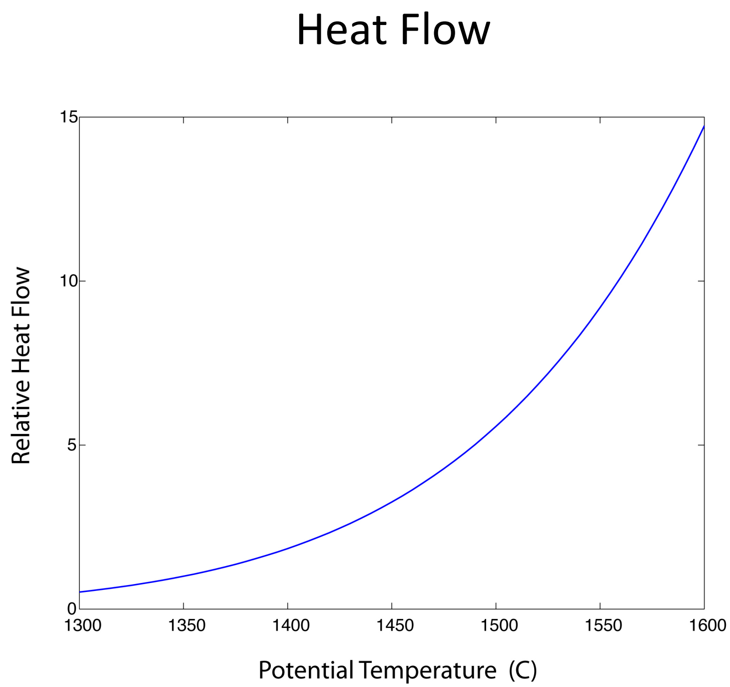
$$Ra(t) = \frac{\alpha g T(t) L^3}{\kappa \nu(t)}$$

Temperature Dependence

$$\nu(T) \propto \exp\left(\frac{E}{RT} \right)$$

Changes in Heat Flow



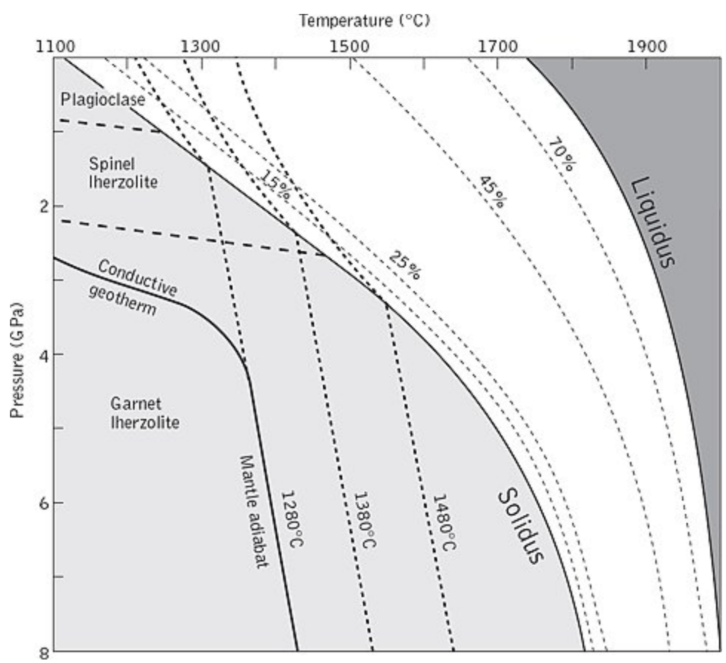


Strong temperature dependence leads to a thermal "catastrophe" at early times

→ argument for high Urey ratio

How is Mantle Convection Different?

Decompression Melting



Melting forms oceanic crust (basalt) and depleted residuum (harzburgite)

Densities

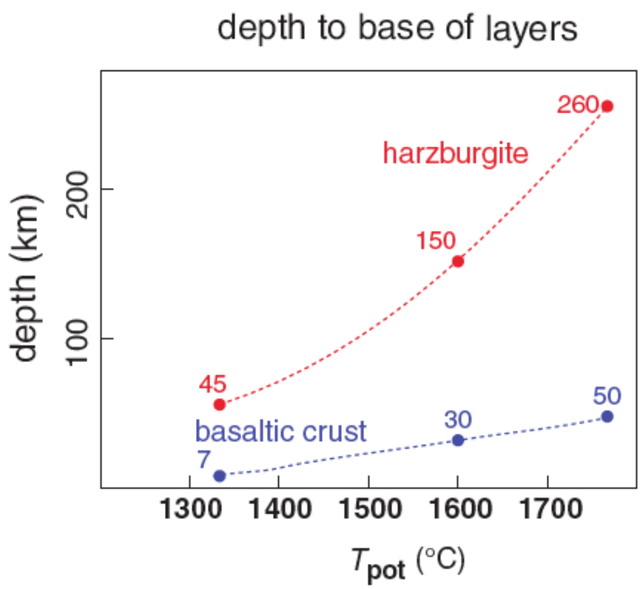
basalt ~ 2.9 g/cm^3

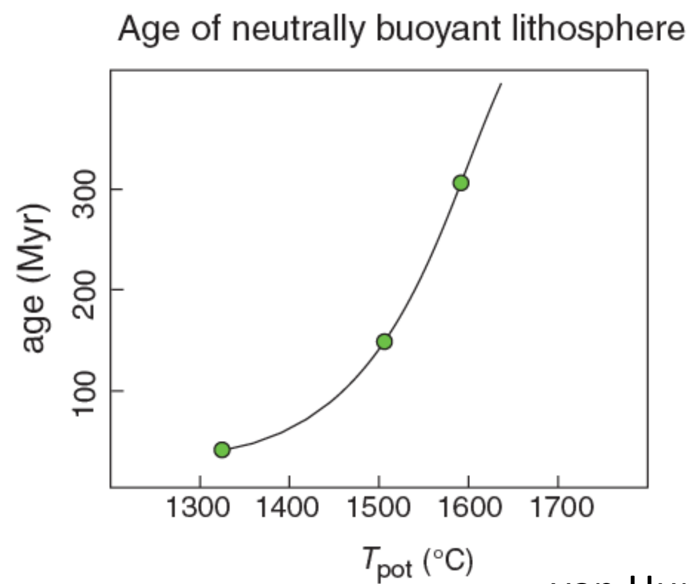
harzburgite ~ 3.2 g/cm^3

lherzolite ~ 3.3 g/cm^3

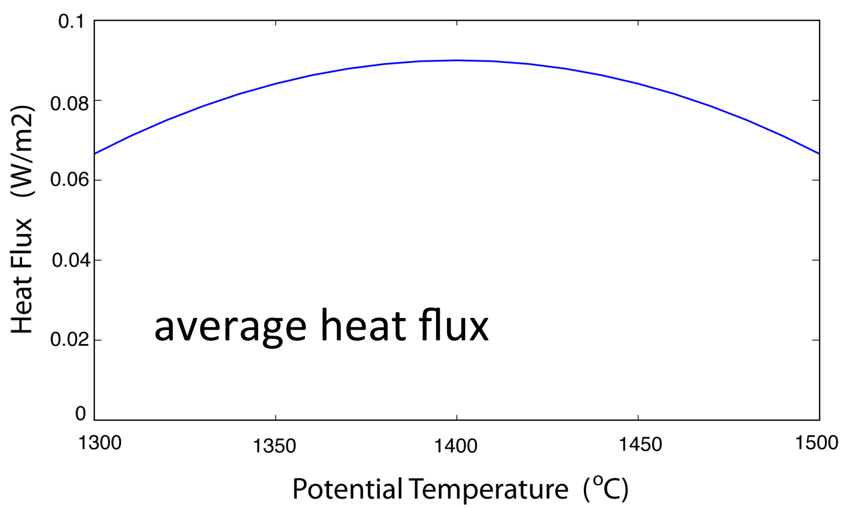
Oxburgh & Parmentier (1977)

Buoyancy of Lithosphere





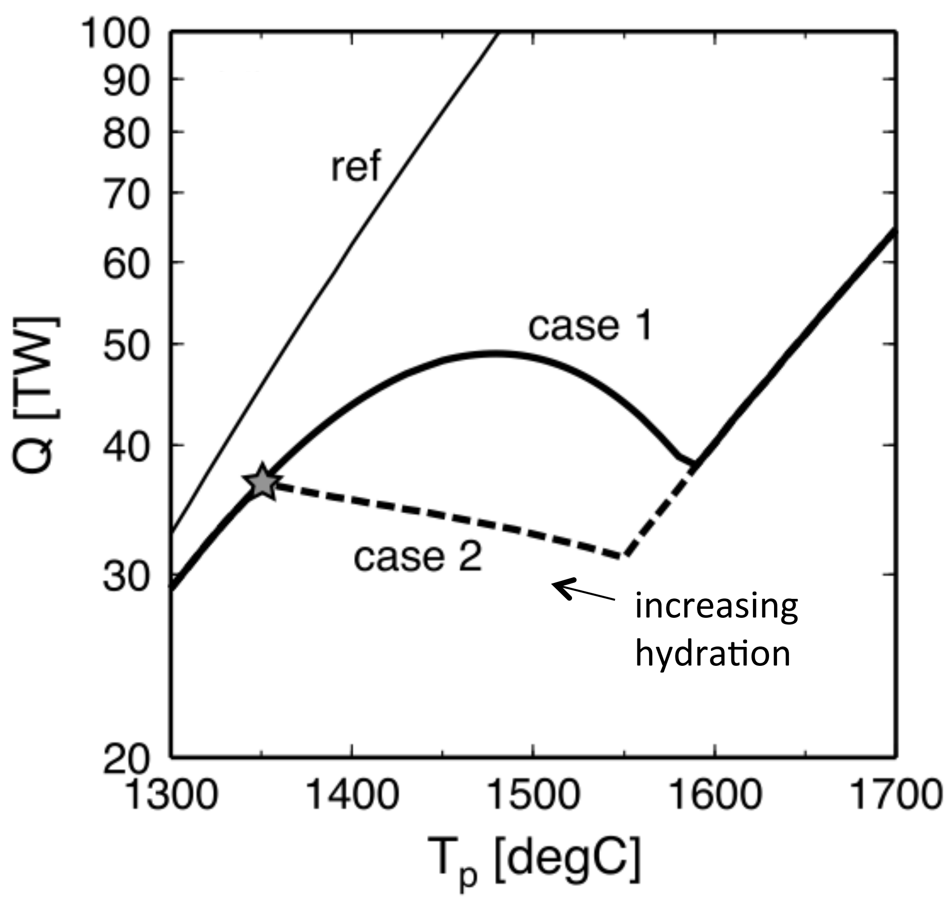
van Hunen et al. (2008)



Sleep (2007)

Rheology of Lithosphere

Melting dehydrates and strengthens the lithosphere (Korenaga, 2010)



Viscous Dissipation includes

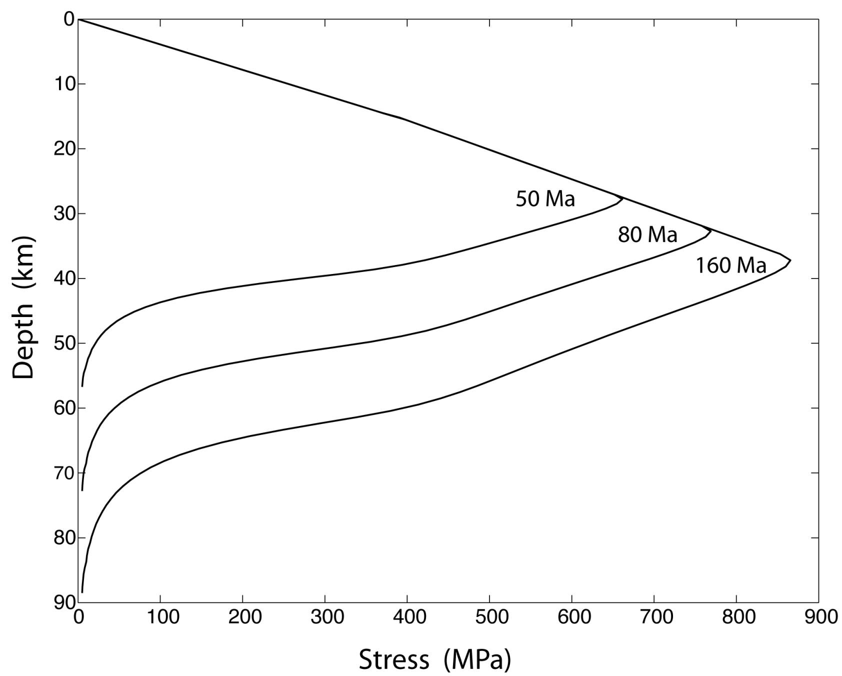
- internal viscosity (↓)
- lithosphere "viscosity" (↑)

$$Nu(t) = \left(\frac{Ra}{2Ra_c}\right)^{1/3} \Delta\eta_L(h)^{-1/3}$$

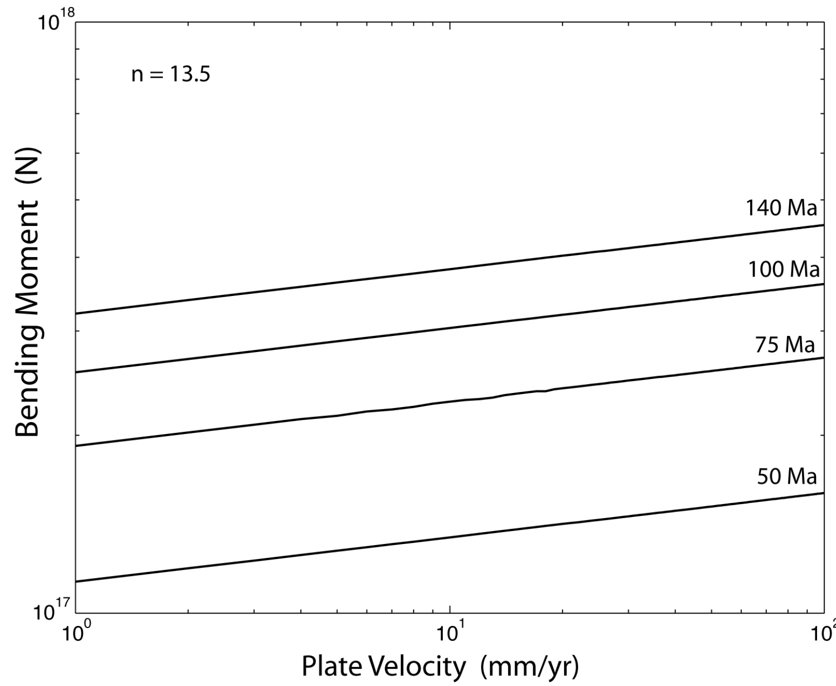
A problem for the magnetic field?

Rheology of Lithosphere

Bending Stress



Bending Moment

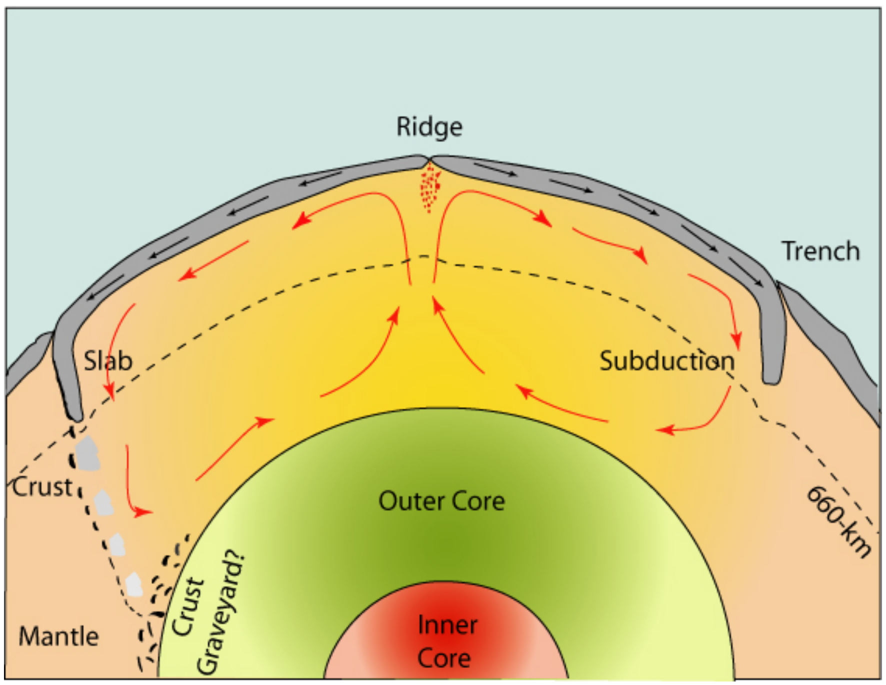


Power law $\dot{\epsilon}_{ij} = \frac{1}{2\eta}(\sigma_{II}^{n-1})\sigma'_{ij}$

(Buffett & Becker, 2012)

Summary

We can make sense of mantle convection using boundary layer theory



(S. Rost)

Extrapolation back in time ?

- heat flow?
- number or size of plates?
- continental configuration?
- surface environment/climate?